# DISTRIBUTIONS AND VON NEUMANN ALGEBRAS OVER FOCK SPACES WITH DEPTH-TWO ACTION

A Dissertation

by

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ABSTRACT

From the standpoint of non-commutative probability, we investigate operators over Fock spaces whose behavior on one level depends only on two of its neighbors. This behavior can be interpreted using the combinatorics of lattice paths and non-crossing partitions. Our first objective is to generalize (via a common framework) the results of Anshelevich (from 2007), Lenczewski & Sałapata (from 2008), and Bożejko & Lytvynov (from 2009), whose constructions exhibited this behavior. We extend a number of results from these papers to our more general setting. These include the quadratic relation satisfied by the generating function for (a variant of) the free cumulants, the resolvent form of the generating function for the Wick polynomials, and classification results for the case when the vacuum state on the operator algebra is tracial. We are able to handle the generating functions in infinitely many variables by considering their matrix-valued versions. Finally, we provide norm estimates guaranteeing that these generating functions are represented by bounded operators.

Our second objective is to focus on a specific class of examples within our framework, which generalizes the free multinomial example from Anshelevich's work. Moreover, their distributions over the Fock space have convolution-power relations with those of the underlying elements of the originating space, seen through the viewpoint of free, Boolean, and (by considering representations with a similar construction) conditionally free cumulants. Moreover, we study the von Neumann algebras generated by these operators given various originating non-commuting probability spaces. In addition, we include the relevant background on free probability, operator algebras, and combinatorics prior to discussing these constructions and results.

# DEDICATION

To my grandparents.

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# 1. INTRODUCTION AND BACKGROUND

## 1.1 Introduction

In quantum mechanics, a Hilbert space serves as the states space of a single particle, but Fock spaces were first introduced by V.A. Fock in 1932 to model the quantum states of multiple identical particles. Informally, a Fock space is the direct sum of Hilbert spaces, each representing zero-particle states, one-particle states, two-particle, and so on. Since the particles are assumed to be identical, each of these $n$-particle state spaces are an $n$-fold tensor product of a single-particle space. If the particles are bosons, the Fock space of symmetric tensor products (see [7]) is the corresponding state space, while the Fock space of anti-symmetric tensor products is the corresponding space for fermions. In addition, we have the Boltzmann (or free) Fock space of the usual tensor products (see Subsection 1.2.11).

Since their introduction, Fock spaces have seen use in a plethora of fields beyond quantum mechanics, including representation theory, combinatorics, and operator algebras. In the last few decades, various Fock spaces which are deformations (some to a greater extent than others) of the classic constructions mentioned above have been studied. With the right choice of operators on such spaces, one can represent various structures, such as commutation relations, orthogonal polynomials, and large classes of probability distributions.

We are particularly interested in three such constructions, because their operators all exhibit the common feature of interaction with only two (adjacent) tensor levels. Compare this to the free Fock space, where no interaction between levels occurs, and the symmetric Fock space, where all levels interact. Our first main objective is to give a general construction of a Fock space with such behavior, over a $*$-probability space $\mathcal{B}$ rather than a Hilbert space, and show that many of the results from each of the three motivating constructions' respective papers hold in this framework.

To be specific, let $\mathcal{B}$ be a unital $*$-algebra, with a positive faithful linear functional $\phi$, and form the algebraic Fock space $\mathcal{F}_{alg}(\mathcal{B}) = \bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$. For each $b \in \mathcal{B}$, we will define operators

$a^+(b), a^-(b), a^0(b)$ and their sum $X(b)$. Here the creation operator $a^+(b)$ is defined in the usual way (see Subsection 1.2.11), but the annihilation operator acts on simple tensors as

$$a^-(b)(u_1 \otimes \ldots \otimes u_n) = (\gamma + \phi)[bu_1]u_2 \otimes \ldots \otimes u_n,$$

where $\gamma + \phi \mathbf{1}_{\mathcal{B}}$ is some completely positive map. Note that this operator couples together the first two components of the tensor. This should be compared with, on the one hand, the Boltzmann (free) Fock space (covered in Subsection 1.2.11), where the annihilation operator acts only on the first component of the tensor; and, on the other hand, with the $q$-Fock space, where its action involves all components of the tensor. Note that if $\gamma = 0$, this is just the standard free annihilation operator (see Subsection 1.2.11) on the full Fock space $\mathcal{F}(L^2(\mathcal{B}, \phi))$. The preservation operator has the form

$$a^0(b)(u_1 \otimes \ldots \otimes u_n) = \Lambda(b)u_1 \otimes u_2 \otimes \ldots \otimes u_n$$

or more generally

$$a^0(b)(u_1 \otimes \ldots \otimes u_n) = \Lambda(b \otimes u_1) \otimes u_2 \otimes \ldots \otimes u_n$$

for a map $\Lambda$ satisfying some symmetry condition, and acts only on the first component of the tensor.

Denoting the 0-length tensor by $\Omega$ (called the vacuum vector), we have an induced vector state $A \mapsto \langle A\Omega, \Omega\rangle$. The behavior of this state over the aforementioned operators $X(b)$ determines their joint distributions, in the sense of free probability. Analyzing these distributions from the perspectives of moments, Boolean cumulants, and free cumulants is our first objective.

On a related topic, free and Boolean cumulant generating functions for free Meixner families [8, 1] satisfy second-order equations. Depth-two action on the Fock space results in such equations being satisfied not by the scalar-valued free cumulant generating function $R(u)$ itself, but by its $\mathcal{B}$-valued "kernel" $R'(u)$ with $R(u) = \phi[uR'(u)u]$. In fact,

$$R'(u)v = v + R'(u)\gamma\,[uR'(u)u]\,v + R'(u)\Lambda(u \otimes v). \tag{1.1}$$

This result easily generalizes to a finite family of variables $\{u_i\}_{i=1}^d$. To make sense of a generating function for joint free cumulants of infinitely many variables $\{u_i\}_{i=1}^\infty$, we take an approach different from [4]. We form an infinite matrix which contains all the information about joint free cumulants of $\{u_i\}_{i=1}^\infty$, and still satisfies (an analogous version of) equation (1.1). In the case when $\Lambda(u \otimes v) = \Lambda(u)v$, for $\{u_i\}_{i=1}^\infty$ uniformly small, this matrix corresponds to a genuine bounded operator. The analysis is similar in style to, but different from, computations with fully matricial free cumulants [9].

In addition, we are also interested in the operators' Wick polynomials, that is, polynomials in $\{X(u) : u \in \mathcal{B}\}$ such that $W(u_1 \otimes u_2 \otimes \ldots \otimes u_n)\Omega = u_1 \otimes u_2 \otimes \ldots \otimes u_n$. We perform a similar matricial analysis for the joint generating function of Wick polynomials. It can be interpreted as an infinite matrix, and (under appropriate assumptions) as a bounded operator. As in [1, 4], it has a resolvent-type form

$$W(u) = (B(u) - X(u))^{-1}(B(u) - \phi[u^2]),$$

where $B(u) = 1 + \Lambda(u) + (\gamma + \phi)[u^2]$. See Section 2.3.

We also look into the algebras generated by all operators $\{X(u) : u \in \mathcal{B}\}$, first by investigating the situation when the vacuum state on the algebra is tracial. The depth-two nature of the action allows us to write down explicit conditions on $\phi, \gamma, \Lambda$ which guarantee this. In the case when $\gamma = 0$, the Fock space is the full Fock space, but the circular operators $X(u)$ are deformed by a non-trivial $\Lambda$. We show that one can always use $\Lambda$ to define a new multiplication on $\mathcal{B}$, so that the representation splits into a semicircular and a free compound Poisson parts. More generally, if $\gamma[u] = \eta u$ for $\eta$ central (related to the construction from [3]) then one has a similar decomposition, but with the third component on which $\Lambda(u \otimes v) = \lambda uv$ for $\lambda$ central.

### 1.1.1 Secondary Construction

The paper [1] contains a special example of a "depth-two action" construction whose algebra generated by the $X(b)$ has a tracial vacuum state, which naturally corresponded to the free multinomial distribution. This example also generalizes to our "primary" setting here, and we find it interesting enough to be considered separately. This construction is a variation of the $t$-deformed free gaussian operators studied in [5] and [6].

As before, we start with a *-algebra $\mathcal{B}$ with a state $\phi$. Denote $\mathcal{B}^\circ = \ker\phi$, and let $\mathcal{T}(\mathcal{B},\phi)$ be the tensor algebra of $\mathcal{B}^\circ$. We denote the element of $\mathcal{T}(\mathcal{B},\phi)$ corresponding to $f \in \mathcal{B}^\circ$ by $X(f)$, so that $\mathcal{T}(\mathcal{B},\phi)$ is identified with the algebra of polynomials in $\{X(f) : f \in \mathcal{B}^\circ\}$. This notation is natural, as we will represent the tensor algebra on two different (families of) Fock spaces, parameterized by a fixed $t \geq 0$. Via these representations, we will define two (families of) states $\Phi_t$ and $\Psi_t$ on $\mathcal{T}(\mathcal{B},\phi)$ (described explicitly in Theorem 3.3.1.1) with the following properties:

- If $\{f_1, \ldots, f_n\} \subset \mathcal{B}^\circ$ are freely independent in $(\mathcal{B},\phi)$, then $\{X(f_1), \ldots, X(f_n)\}$ are freely independent with respect to $\Phi_t$.
- If $\{f_1, \ldots, f_n\} \subset \mathcal{B}^\circ$ are Boolean independent in $(\mathcal{B},\phi)$, then $\{X(f_1), \ldots, X(f_n)\}$ are freely independent with respect to $\Psi_t$ and conditionally free with respect to the pair $(\Phi_t, \Psi_t)$.

We are also interested in the study of the von Neumann algebras $W^*(\mathcal{T}(\mathcal{B},\phi), \Phi_t)$ arising in the GNS representation of $\mathcal{T}(\mathcal{B}^\circ)$ with respect to $\Phi_t$. To summarize:

- Let $\mathcal{B} = \underset{\alpha}{\mathbb{C}} \oplus \underset{1-\alpha}{\mathbb{C}}$, with the state determined by the parameter $\alpha \in (0, \frac{1}{2}]$. Then

$$W^*\left(\mathcal{T}\left(\underset{\alpha}{\mathbb{C}} \oplus \underset{1-\alpha}{\mathbb{C}}, \phi\right), \Phi_t\right) \simeq \begin{cases} \underset{t}{L^\infty[0,1]} \oplus \underset{1-\alpha-\alpha t}{\mathbb{C}} \oplus \underset{\alpha-(1-\alpha)t}{\mathbb{C}}, & t < \frac{\alpha}{1-\alpha} \\ \underset{\alpha+\alpha t}{L^\infty[0,1]} \oplus \underset{1-\alpha-\alpha t}{\mathbb{C}}, & \frac{\alpha}{1-\alpha} \leq t < \frac{1-\alpha}{\alpha} \\ L^\infty[0,1], & \frac{1-\alpha}{\alpha} \leq t \end{cases}$$

- Let $\mathcal{B}$ be the free product $*_{i=1}^d \left(\underset{\alpha_i}{\mathbb{C}} \oplus \underset{1-\alpha_i}{\mathbb{C}}\right)$, $p_i^\circ$ the projection with trace $\alpha_i$ in the $i$th copy of

$\mathbb{C}^2$, and assuming without loss of generality that the $\alpha_i \subset (0, \frac{1}{2}]$ are increasing. Then

$$W^*(X(p_i^\circ) : 1 \leq i \leq d) \simeq \begin{cases} \underset{1-\gamma_1-\gamma_2}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2}{\mathbb{C}}, & t < \frac{\alpha_d - \sum_{i=1}^{d-1}\alpha_i}{1-\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)}, \\ \underset{1-\gamma_1}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_1}{\mathbb{C}}, & \frac{\alpha_d - \sum_{i=1}^{d-1}\alpha_i}{1-\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)} \leq t < \frac{1-\left(\left(\sum_{i=1}^{d}\alpha_i\right)\right)}{\left(\sum_{i=1}^{d}\alpha_i\right)}, \\ \mathbb{F}_d, & \frac{1-\left(\left(\sum_{i=1}^{d}\alpha_i\right)\right)}{\left(\sum_{i=1}^{d}\alpha_i\right)} \leq t, \end{cases}$$

where

$$\gamma_1 = \max\left\{1 - \left(\sum_{i=1}^{d}\alpha_1\right)(1+t), 0\right\}$$
$$\gamma_2 = \max\left\{\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)(1+t) - t, 0\right\}$$

and $x$ is chosen so that the free dimension is the sum of the free dimensions of $W^*(X(p_i^\circ))$.

- For $1 \leq i \leq d$, let $p_i^\circ = \frac{1}{2}(E_{1,1+i} + E_{1+i,1}) \in M_{d+1}(\mathbb{C})$ with the trace $\phi_{11}$ which returns the $1,1$ entry of a matrix. Then

$$W^*(X(p_i^\circ) : 1 \leq i \leq d) \simeq \begin{cases} \mathcal{L}(\mathbb{F}_d) & \text{if } t \geq \sqrt{d} \\ \mathcal{L}(\mathbb{F}_d) \oplus \mathbb{B}(\ell_2) & \text{otherwise.} \end{cases}$$

- Let $\mathcal{B} = L^\infty[0,1]$, with the state given by integrating with respect to the Lebesgue measure. Then for $t > 0$, $W^*(\mathcal{T}(L^\infty[0,1], dx), \Phi_t)$ is a $\mathrm{II}_1$-factor.

### 1.1.2 Organization of This Work

The background section is meant to serve as a convenient reference, as well as introduce one who is familiar with operator algebras but not free probability theory, to the subject. Subsections 1.2.1, 1.2.2, and 1.2.5 establish the basic concepts of free probability theory (taking care to define terms whose precise meanings vary from author to author), while 1.2.3 and 1.2.4 cover the relevant basics of von Neumann algebras and how free probability can help us understand them. Free cumulants, relevant combinatorics, and the related free additive convolution are covered in 1.2.6 and 1.2.7. Then subsections 1.2.8 and 1.2.9 summarize the analogous concepts of conditionally free probability and operator-valued probability. In Subsection 1.2.10, the relevant combinatorics of lattice paths are summarized, along with their relations to non-crossing partitions. Finally, a basic Fock space model is given as an example, on which the operators of interest have the semi-circle distribution. This also serves as a natural starting point to discuss the ways our construction generalizes this and the three motivating constructions mentioned before.

The first phase begins in Section 2.1, where we first present our main construction. In Section 2.2, we prove formulas for joint moments, and Boolean and free cumulants, of the operators $\{X(u) : u \in \mathcal{B}\}$. We also compare and contrast these formulas with the operator-valued results of [10]. In particular, unlike in [10], the inner product in this paper is scalar-valued rather than $\mathcal{B}$-valued. In Section 2.3, we discuss Wick polynomials, and matricial generating functions for them and for the free cumulants. In Section 2.4, we provide conditions under which operators $X(u)$, as well as various generating functions, are bounded. In Section 2.5 we derive the conditions for the vacuum state to be tracial. We also prove a representation theorem under the assumption that the vacuum state is tracial. The results in Sections 2.3 and 2.4 are proven in the setting of $\Lambda(u \otimes v) = \Lambda(u)v$; in contrast, the results in Section 2.5 are of main interest for general $\Lambda$.

The secondary construction can be found in Section 3.1. We also define yet another construction, for the purpose of creating a second state, with which we will study conditional freeness of our operators. We then discuss some previous works which have influenced our own. In Section 3.2, we present the relationships between the distributions of the underlying $f \in \mathcal{B}$, and their counterparts $X(f)$. In particular, we discuss the implications of free or Boolean independence in $\mathcal{B}$ on the corresponding $X(f)$ with respect to the main vacuum state or the second state. In Section 3.3, we study the von Neumann algebras generated by certain $X(f)$ with respect to those states, for several starting algebras $\mathcal{B}$.

## 1.2 Background

### 1.2.1 Free Probability and Operator Algebra Terminology

In free probability, the primary object of interest is a *(non-commutative) probability space (NCPS)*, which is a pair $(\mathcal{A}, \phi)$ of the following:

- a $*$-algebra $\mathcal{A}$ that is *unital* (contains a multiplicative unit $1 \in \mathcal{A}$), and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$, which is a

    *state*, which simply means it is a *unital* map: $\phi[1] = 1$. It is also

    *self-adjoint:* $\phi[a^*] = \overline{\phi[a]}$, and

    *positive:* $\phi[a^*a] \geq 0$.

Elements of the space are called *non-commuting random variables*.

A word of caution is in order: some authors define a NCPS without the $*$-operation, in which case the last two properties of $\phi$ need not hold. These authors usually call the space defined above a $*$-*probability space* ($*$-*PS*) instead.

In applications to operator algebras especially, $\mathcal{A}$ usually comes with more structure. First, NCPSs are often subalgebras of the algebra of bounded linear operators over some Hilbert space $\mathcal{H}$, denoted $B(\mathcal{H})$. On this algebra, we have the *operator norm*:

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\|=1} \frac{\|x\xi\|}{\|\xi\|}.$$

Any vector *-subalgebra of $B(\mathcal{H})$ that is closed under this norm is called a *C*-algebra*.* For a $*$-PS $(\mathcal{A}, \phi) \subset B(\mathcal{H})$, if $\phi$ is norm-continuous, the space is called a *C*-probability space (C*-PS)*.

Even more structure can be imposed, via two more topologies that need to be defined- using convergence to define these is far more useful and intuitive (for our purposes) than to describe their

*Alternatively, one can define a C*-algebra as a Banach $*$-algebra over $\mathbb{C}$ whose norm satisfies the *C* identity*: $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. We will discuss this further in 1.2.3.1.

bases. A net $\{x_i\} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the *strong operator topology (SOT)* if

$$\lim_i \|(x - x_i)\xi\| = 0 \ \forall \xi \in \mathcal{H}.$$

This is the topology of point-wise convergence, translated to the context of linear operators. Similarly, a net $\{x_i\} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in the *weak operator topology (WOT)* if

$$\lim_i \langle (x - x_i)\xi, \eta\rangle = 0 \ \forall \xi, \eta \in \mathcal{H}.$$

For a subset $X \subset B(\mathcal{H})$, its *commutant* is

$$X' := \{y \in B(\mathcal{H}) \,|\, xy = yx \ \forall x \in X\}.$$

Its *double commutant* is $X'' := (X')'$.

**Theorem 1.2.1.1.** ***(von Neumann, Bicommutant Theorem, 1929)*** *For any unital $*$-subalgebra $M \subset B(\mathcal{H})$,*

$$\overline{M}^{SOT} = \overline{M}^{WOT} = M'',$$

*where $\overline{M}^{SOT}$ and $\overline{M}^{WOT}$ denote the closures of $M$ under the strong and weak operator topologies, respectively. Thus, these analytic constructions and the purely algebraic double commutant are, in fact, the same object, called a von Neumann algebra.*

One last topology that needs to be defined on $M$ is the *$\sigma$-weak topology*: a net $\{x_i\} \subset B(\mathcal{H})$ converges to $x \in B(\mathcal{H})$ in this topology if

$$\lim_i \sum_{n=1}^{\infty} \langle (x - x_i)\xi_n, \eta_n\rangle = 0 \ \forall \text{ sequences } \{\xi_n\}_{n\in\mathbb{N}}, \{\eta_n\}_{n\in\mathbb{N}} \subset \mathcal{H} \text{ such that } \sum_n \|\xi_n\|^2, \sum_n \|\eta_n\|^2 < \infty.$$

A state $\psi$ is *normal* if it is $\sigma$-weakly continuous. For a $*$-PS $(\mathcal{A}, \phi) \subset B(\mathcal{H})$, if $\mathcal{A}$ is a von Neumann algebra and $\phi$ is normal, the space is called a *W*-probability space (W*-PS)*.

A state $\psi : M \to \mathbb{C}$ is *faithful* if $\psi[x^*x] = 0$ only if $x = 0$, while a state $\psi$ on a NCPS $\mathcal{A}$ is *tracial*, or a *trace*, if for all $x, y \in \mathcal{A}$, $\psi[xy] = \psi[yx]$.

Finally, let us discuss properties of maps between $*$-algebras $\mathcal{B}$ and $\mathcal{C}$. An element $b \in \mathcal{B}$ is *positive* if $b = \sum_{i=1}^{k} u_i^* u_i$ for some $k$ and $u_i \in \mathcal{B}$. A map $T : \mathcal{B} \to \mathcal{C}$ is *positive* if for each $u \in \mathcal{B}$, $T(u^*u)$ is positive in $\mathcal{C}$. It is *completely positive* if for each $n$, the map

$$T_n : M_n(\mathcal{B}) \to M_n(\mathcal{C}), \;\; T_n\left([a_{ij}]_{i,j=1}^n\right) = [T(a_{ij})]_{i,j=1}^n$$

is positive, where we use the usual $*$-structure on $M_n(\mathcal{B})$. It is *faithful* if $T(u^*u) = 0$ only for $u = 0$.

Another inconsistency between authors should be noted: many restrict the definition of non-commuting random variable to only the *self-adjoint* elements, that is, those that satisfy $a = a^*$, while others consider all elements. This isn't an issue most of the time, due to the fact that every element of a $*$-algebra can be written as a linear combination of two self-adjoint elements:

$$a = \Re a + i \Im a \quad \text{where} \quad \Re a := \frac{1}{2}(a + a^*) \quad \text{and} \quad \Im a = \frac{1}{2i}(a - a^*).$$

From here on, we will focus our attention on self-adjoint operators unless noted otherwise.

*1.2.1.1 Key Examples*

**Example 1.2.1.2.** Let $(\Omega, \Sigma, P)$ be a probability measure space. Then

$$\mathcal{A} = L^\infty(\Omega, P) \text{ (with } f^* = \overline{f}) \text{ paired with expectation } \mathbb{E}[f] = \int_\Omega f dP$$

is a commutative probability space. Moreover, $\mathcal{A}$ is a von Neumann algebra, and even more is true:

**Theorem 1.2.1.3.** ***(Gelfand, Naimark)*** *If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a commutative von Neumann algebra, then it is $*$-isomorphic to $L^\infty(X, \mu)$ for some measure space $(X, \Sigma, \mu)$, where $X$ is a compact Hausdorff space and $\mu$ is a positive, regular Borel measure.*

Note that the unital condition on any expectation $\phi$ implies all commutative probability spaces under the earlier definitions must take the form of the preceding example.

**Example 1.2.1.4.** Extending the previous example, let

$$\mathcal{A} = M_n(\mathbb{C}) \otimes L^\infty(\Omega, P).$$

These are *random matrices*. This is a NCPS when paired with the state

$$\phi[A] = \frac{1}{n}(\mathrm{Tr} \otimes \mathbb{E})(A) = \frac{1}{n}\mathrm{Tr}\left(\left[\int_\Omega A_{ij} dP\right]_{ij}\right).$$

In other words, take the normalized trace of the matrix whose entries are the expected values of their respective entries from the random matrix $A$.

The following example is the subject of a famous (still) open problem in operator theory, and was a major inspiration of the notions of free independence and free products, which will be discussed later.

**Example 1.2.1.5.** For a discrete group $G$, let $\mathbb{C}[G]$ denote the space of finite linear combinations of elements of $G$ over $\mathbb{C}$. This is a vector algebra with multiplication

$$\left(\sum_{g\in G} \alpha_x x\right)\left(\sum_{h\in G} \beta_h h\right) = \sum_{g\in G} \alpha_g \beta_{g^{-1}h} h.$$

These vectors can also be considered finitely supported functions $G \to \mathbb{C}$, where it is clear that this multiplication is just convolution. With the $*$-operation

$$f^*(g) = \overline{f(g^{-1})},$$

$\mathbb{C}[G]$ is a $*$-algebra with unit $\delta_e = 1e$, where $e \in G$ is the group's identity.

Finally, define the expectation

$$\tau[f] = f(e),$$

or, in terms of linear combinations, this map simply returns the coefficient assigned to $e$.

### 1.2.2 Distributions and Free Independence

Free probability is so named due to its abundance of analogies with classical probability theory, starting with the basic concepts of this section.

**Definition 1.2.2.1.** For $x \in (\mathcal{A}, \phi)$, a symmetric element of a NCPS, its *$n$th moment* (with respect to $\phi$) is the number $\phi[x^n]$ ($n \in \mathbb{N}$). Its first moment is called its *mean*. These numbers uniquely determine its *distribution (with respect to $\phi$)*, the linear functional

$$\phi_x : \mathbb{C}[X] \to \mathbb{C}, \qquad \phi_x[p(X)] = \phi[p(x)].$$

**Definition 1.2.2.2.** More generally, for $x_1, ..., x_n \in (\mathcal{A}, \phi)$ symmetric, their *joint (or mixed) moments* are the numbers

$$\{\phi[x_{u(1)}x_{u(2)}...x_{u(k)}] \,|\, k \geq 0,\ 1 \leq u(i) \leq n\},$$

which uniquely define their *joint distribution*

$$\phi_{x_1,...,x_n} : \mathbb{C}\langle X_1, ..., X_n\rangle \to \mathbb{C}, \qquad \phi_{x_1,...,x_n}[p(X_1, ..., X_n)] = \phi[p(x_1, ..., x_n)].$$

Note that every distribution (resp. joint distribution) can be considered an expectation map on the *-algebra $\mathbb{C}[X]$ (resp. $\mathbb{C}\langle X_1, ..., X_n\rangle$), with $*$-operation

$$x_i^* = x_i,$$

$$(x_{u(1)}x_{u(2)}...x_{u(k)})^* = x_{u(k)}...x_{u(2)}x_{u(1)}.$$

Recall that in classical probability, two random variables $X : \Omega \to \mathbb{C}$ and $Y : \Omega \to \mathbb{C}$ are independent if and only if $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$ for all bounded continuous functions

$f, g : \mathbb{C} \to \mathbb{C}$. The generalization of this idea to a non-commutative probability space $(\mathcal{A}, \phi)$ serves as the definition of independence in this context:

**Definition 1.2.2.3.** Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n \subset (\mathcal{A}, \phi)$ are *(classically) independent* with respect to $\phi$ if

- $ab = ba$ for any $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$ for $i \neq j$, and
- For all $a_i \in \mathcal{A}_i$,

$$\phi[a_1 a_2 ... a_n] = \phi[a_1]\phi[a_2]...\phi[a_n].$$

This definition is quite restrictive, since in most cases, the subalgebras we wish to study do not usually commute with each other. A much more useful notion of independence, one that does not require commutativity of any degree, is as follows:

**Definition 1.2.2.4.** (**Voiculescu**) Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n \subset (\mathcal{A}, \phi)$ are *freely independent*, or *free*, with respect to $\phi$ if whenever

a. $a_i \in \mathcal{A}_{u(i)}$ such that $u(1) \neq u(2),\ u(2) \neq u(3),\ ...,\ u(n-1) \neq u(n)$, and

b. $\phi[a_i] = 0\ \forall i = 1, ..., n$, we have

$$\phi[a_1 a_2 ... a_n] = 0.$$

In other words, any product of mean-zero terms, where no consecutive pair of terms in the product belongs to the same subalgebra, has mean zero.

A third form of independence often appears in non-commutative spaces:

**Definition 1.2.2.5.** Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n \subset (\mathcal{A}, \phi)$ that do not contain the unit[†] of $\mathcal{A}$ are *Boolean independent* with respect to $\phi$ if whenever

$$a_i \in \mathcal{A}_{u(i)} \text{ such that } u(1) \neq u(2),\ u(2) \neq u(3),\ ...,\ u(n-1) \neq u(n),$$

[†]The definition would yield a degenerate structure otherwise. For example, if $\mathcal{A}_1$ contained the unit, then for $a, b \in \mathcal{A}_2$, we would have $\phi[a1b] = \phi[a]\phi[1]\phi[b]$ by definition, but the left-hand side is $\phi[ab]$, while the right-hand side is $\phi[a]\phi[b]$.

$$\phi[a_1 a_2 ... a_n] = \phi[a_1]\phi[a_2]...\phi[a_n].$$

In other words, any product of terms (with possibly non-zero mean), where no consecutive pair of terms in the product belongs to the same subalgebra, has mean equal to the product of the terms' individual means.

Elements are called independent (resp. freely independent) if the *-subalgebras they generate with the unit of $\mathcal{A}$ are independent (resp. freely independent). Elements are Boolean independent if the *-subalgebras they generate (without the unit of $\mathcal{A}$) are. Finally, we end this subsection with other important definitions regarding distributions.

**Definition 1.2.2.6.** A set of variables $\{a_i\}_{i\in I} \subset \mathcal{A}$ are *identically distributed* with respect to $\phi$ if

$$\phi[a_i^n] = \phi[a_j^n] \text{ for all } n \in \mathbb{N} \text{ and } i, j \in I.$$

**Definition 1.2.2.7.** Let $\{(\mathcal{A}_k, \phi_k)\}_{k\in\mathbb{N}}$ and $(\mathcal{A}, \phi)$ be non-commutative probability spaces.

- Let $b_k \in \mathcal{A}_k$, $b \in \mathcal{A}$. We say $\{b_k\}$ *converges in distribution* to $b$ if

$$\lim_{k\to\infty} \phi_k[b_k^n] = \phi[b^n] \text{ for all } n \in \mathbb{N}.$$

- More generally, for an index set $I$, let $b_k^{(i)} \in \mathcal{A}_k$ for $k \in \mathbb{N}$ and each $i \in I$, and let $b^{(i)} \in \mathcal{A}$ for each $i \in I$. We say that $\{b_k^{(i)}\}_{i\in I}$ *converges in distribution* to $b^{(i)}$ if

$$\lim_{k\to\infty} \phi_k[b_k^{(i_1)}...b_k^{(i_n)}] = \phi[b^{(i_1)}...b^{(i_1)}] \text{ for all } n \in \mathbb{N} \text{ and } i_1, ..., i_n \in I.$$

### 1.2.3 von Neumann Algebras

#### *1.2.3.1 Abstractly Defined *-Algebras*

Often, C*-algebras are considered independently of a Hilbert space under the following definitions: let $\mathcal{A}$ be a normed algebra over $\mathbb{C}$ that is norm-complete and satisfies $\|ab\| \leq \|a\|\|b\|$ for all

$a, b \in \mathcal{A}$ (that is, a *Banach algebra*). A map $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ is called an *involution* if

- $(a^*)^* = a$
- $(a+b)^* = a^* + b^*$
- $(xy)^* = y^* x^*$
- $(\lambda x)^* = \overline{\lambda} x^*$ for $\lambda \in \mathbb{C}$.
- (Some authors also require the involution to be *isometric*: $\|x^*\| = \|x\|$. We will require it as well.)

An element $a \in \mathcal{A}$ is called *symmetric*[‡] if $a^* = a$.

A Banach algebra with an involution * is called a *C*-algebra* if it satisfies the *C* property*: $\|a^* a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

A *W*-algebra* $\mathcal{M}$ is a C*-algebra that, when considered as a Banach space, has a pre-dual space, that is, there exists a Banach space $X$ such that $X^* = \mathcal{M}$. The *$\sigma$-topology* on it is the weak-* topology on $\mathcal{M}$ generated by $X$ as bounded linear functionals on it.

A $*$-homomorphism $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ from one W*-algebra into another is said to be a *W*-homomorphism* if it is $\sigma$-to-$\sigma$ continuous. Moreover, the image of this mapping is always $\sigma$-closed (Proposition 1.16.2, [11]).

A *$*$-representation* of a C*-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, while a *W*-representation* of a W*-algebra $\mathcal{M}$ is a W*-homomorphism $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

*1.2.3.2 Unitization*

If $\mathcal{A}$ is a non-unital (abstract) C*-algebra, we can "add" a unit to make it unital; that is, we embed $\mathcal{A}$ into the smallest unital C*-algebra containing it. To be more precise, let

$$\tilde{\mathcal{A}} := \mathcal{A} \oplus \mathbb{C},$$

[‡]This is more general terminology than self-adjoint, which refers to bounded operators on a Hilbert space.

with the following algebraic operations:

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta)$$
$$(a,\alpha)^* = (a^*, \overline{\alpha})$$
$$\|(a,\alpha)\| = \sup_{b\in\mathcal{A},\|b\|=1} \|ab + \alpha b\|.$$

This definition comes from the concrete case in $\mathcal{B}(\mathcal{H})$: If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is non-unital, we can take the C*-algebra generated by $\mathcal{A}$ and the identity operator on $\mathcal{H}$. In this case, we can justify writing elements of this C*-algebra as $a + \alpha 1$, and most authors (myself included) do so in the abstract case as well, rather than writing $(a, \alpha)$.

*1.2.3.3 The GNS Construction*

Now suppose we have a W*-algebra $\mathcal{M}$ along with some faithful, $\sigma$-continuous state $\phi : \mathcal{M} \to \mathbb{C}$ (a state on a non-unital C*-algebra is a linear functional with norm 1). Then it turns out that we get a von Neumann algebra:

**Theorem 1.2.3.1.** ***(Gelfand, Naimark, Segal (GNS) Construction)*** *Suppose we have a W*-algebra $\mathcal{M}$ along with some faithful, $\sigma$-continuous state $\phi : \mathcal{M} \to \mathbb{C}$. Then there exists a Hilbert space $\mathcal{H}$, an injective W*-representation $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$, and a cyclic vector $\xi \in \mathcal{H}$ such that*

$$\phi(x) = \langle \pi(x)\xi, \xi\rangle \ \forall x \in \mathcal{M}.$$

*In particular, $\mathcal{M}$ is isomorphic to its image through this map.*

*Proof.* Without loss of generality (by considering unitizations), assume that $\mathcal{M}$ is unital. Define the sesquilinear form

$$\langle x, y\rangle_\phi = \phi(y^*x).$$

By faithfulness, it is a (non-degenerate) inner product on $\mathcal{M}$. By taking the closure with respect to this inner product, we get a Hilbert space $\mathcal{H}_\phi$, with norm induced by the inner product.

Next, denoting by $\Lambda : \mathcal{M} \to \mathcal{H}_\phi$ the natural embedding and $\mathcal{M}_\phi := \Lambda(\mathcal{M})$, we can define the representation $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H}_\phi)$ by

$$\pi(x)\Lambda(y) = \Lambda(xy).$$

Through density of $\mathcal{M}_\phi$ in $\mathcal{H}_\phi$, it suffices to check that this is a *-homomorphism through its behavior on it:

$$\pi(x)\pi(y)\Lambda(z) = \Lambda(xyz) = \pi(xy)\Lambda(z),$$

$$\begin{aligned}\langle \pi(x)^*\Lambda(y), \Lambda(z)\rangle &= \langle \Lambda(y), \pi(x)\Lambda(z)\rangle = \langle \Lambda(y), \Lambda(xz)\rangle \\ &= \phi((xz)^*y) = \phi(z^*x^*y) = \langle \Lambda(x^*y), \Lambda(z)\rangle = \langle \pi(x^*)\Lambda(y), \Lambda(z)\rangle.\end{aligned}$$

Next, to verify faithfulness, assume $\pi(x) = 0$. Then

$$0 = \pi(x)^*\pi(x) = \pi(x^*x), \text{ so}$$

$$0 = \langle \pi(x^*x)\Lambda(1), \Lambda(1)\rangle = \langle \Lambda(x^*x), \Lambda(1)\rangle = \phi(x^*x),$$

so by faithfulness of $\phi$, $x = 0$.

Next,

$$\|\pi(x)\Lambda(y)\|_2^2 = \|\Lambda(xy)\|_2^2 = \phi((xy)^*(xy)) = \phi(y^*x^*xy).$$

We know that $y^*x^*xy \le \|x\|^2 y^*y$. Since $\phi$ is positive, $\phi(y^*x^*xy) \le \|x\|^2\phi(y^*y) = \|x\|^2\|\Lambda(y)\|^2$. Hence, $\pi(x) \in \mathcal{B}(\mathcal{H})$.

Let $\xi = \Lambda(1)$, which is clearly cyclic for $\mathcal{M}$. Also, for $x \in \mathcal{M}$,

$$\phi(x) = \phi(1^*x) = \langle \Lambda(x), \Lambda(1)\rangle = \langle \pi(x)\Lambda(1), \Lambda(1)\rangle = \langle \pi(x)\xi, \xi\rangle.$$

All that remains is to check that $\pi$ is normal (This follows the remarks given on page 41 of

[11]).

First, we'll need the following proposition (Corollary 1.15.5 from [11]): A linear functional $f$ on $\mathcal{B}(\mathcal{H})$ is in the predual of $\mathcal{B}(\mathcal{H})$ if and only if there exist sequences $\{\xi_n\}$, $\{\eta_n\} \subset \mathcal{H}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty, \text{ and } f(x) = \sum_{n=1}^{\infty} \langle x\xi_n, \eta_n \rangle.$$

Using this for an arbitrary $f \in$ the predual of $\mathcal{B}(\mathcal{H}_\phi)$, we get $\{\xi_n\}, \{\eta_n\} \subset \mathcal{H}$ as described in the proposition. Since $\mathcal{M}_\phi$ is dense in $\mathcal{H}_\phi$, there exist families of sequences $\{a_{n,m}\}_m, \{b_{n,m}\}_m \subset \mathcal{M}_\phi$ such that $\|a_{n,m} - \xi_n\| \to 0$ and $\|b_{n,m} - \eta_n\| \to 0$ as $m \to \infty$ for all $n \in \mathbb{N}$. Moreover, they can be chosen so that $\|\xi_n - a_{n,m}\| < \frac{1}{2^{n/2}}$ and $\|\eta_n - b_{n,m}\| < \frac{1}{2^{n/2}}$ for all $n$. Then

$$\begin{aligned}
&\left| \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \eta_n \rangle - \sum_{n=1}^{\infty} \langle \pi(x) a_{n,m}, b_{n,m} \rangle \right| \\
&\leq \left| \sum_{n=1}^{\infty} \langle \pi(x)(\xi_n - a_{n,m}), \eta_n \rangle \right| + \left| \sum_{n=1}^{\infty} \langle \pi(x) a_{n,m}, (\eta_n - b_{n,m}) \rangle \right| \\
&\leq \|x\| \sum_{n=1}^{\infty} \|\xi_n - a_{n,m}\| \|\eta_n\| + \|x\| \sum_{n=1}^{\infty} \|a_{n,m}\| \|\eta_n - b_{n,m}\| \\
&\leq \|x\| \left( \sum_{n=1}^{\infty} \|\xi_n - a_{n,m}\|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \|\eta_n\|^2 \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\|x\|\left(\sum_{n=1}^{\infty}\|a_{n,m}\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|\eta_n-b_{n,m}\|^2\right)^{1/2}\\
\leq\ &\|x\|\left(\sum_{n=1}^{\infty}\|\xi_n-a_{n,m}\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|\eta_n\|^2\right)^{1/2}\\
&+\|x\|\left(\left(\sum_{n=1}^{\infty}\|a_{n,m}-\xi_n\|^2\right)^{1/2}+\left(\sum_{n=1}^{\infty}\|\xi_n\|^2\right)^{1/2}\right)\left(\sum_{n=1}^{\infty}\|\eta_n-b_{n,m}\|^2\right)^{1/2}\\
\leq\ &\|x\|\left(\sum_{n=1}^{\infty}\|\xi_n-a_{n,m}\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|\eta_n\|^2\right)^{1/2}\\
&+\|x\|\left(1+\left(\sum_{n=1}^{\infty}\|\xi_n\|^2\right)^{1/2}\right)\left(\sum_{n=1}^{\infty}\|\eta_n-b_{n,m}\|^2\right)^{1/2}
\end{aligned}
$$

This converges to zero by the dominated convergence theorem (when considering the difference sums as integrals over counting measure), since the sequences $\{\|\xi_n-a_{n,m}\|^2\}$ and $\{\|\eta_n-b_{n,m}\|^2\}$ are dominated above by $\frac{1}{2^n}$ by choice of sequences.

Hence, $f(x)$ is a uniform limit of sequences $\{f_m(x)\}$ on the unit sphere of $\mathcal{M}$, where

$$f_m(x)=\sum_{n=1}^{\infty}\langle\pi(x)a_{n,m},b_{n,m}\rangle=\sum_{n=1}^{\infty}\phi(b_{n,m}^*xa_{n,m}).$$

$f\circ\pi\in$ the predual of $\mathcal{M}$ (when considered as a subspace of $\mathcal{M}^*$, its dual space), since $f_m\in$ the predual. Hence, the mapping $x\mapsto\pi(x)$ is normal from bounded spheres in $\mathcal{M}$ into $\mathcal{B}(\mathcal{H}_\phi)$, so it is normal on all of $\mathcal{M}$. □

**Proposition 1.2.3.2.** *The GNS construction is unique in the sense that if $\mathcal{H}'$ is another Hilbert space such that there exists an injective W*-representation $\pi':\mathcal{M}\to\mathcal{B}(\mathcal{H}')$, with cyclic vector $\xi'$ such that $\phi(x)=\langle\pi(x)\xi',\xi'\rangle$, then $\pi'(\mathcal{M})$ is isomorphic to the image $\pi(\mathcal{M})$ previously given.*

All of this shows that every W*-algebra paired with a faithful normal state is a von Neumann algebra. This justifies our restriction of the definition of W*-probability space to von Neumann subalgebras of bounded operators over a Hilbert space paired with a normal expectation map. (If the state is not faithful, then we replace $\mathcal{H}_\phi$ in the GNS construction with its quotient space induced by $\{x \in \mathcal{M} | \phi(x^*x) = 0\}$, a situation I've omitted from this work.) Note that some authors refer to W*-algebras as abstract von Neumann algebras, while their GNS constructions are called concrete.

*1.2.3.4 The Spectral Theorem*

**Definition 1.2.3.3.** For a set $X$, a $\sigma$-algebra $\Sigma$ of subsets of $X$, and a Hilbert space $\mathcal{H}$, a *spectral measure* on $(X, \Sigma, \mathcal{H})$ is a function $E : \Sigma \to \mathcal{B}(\mathcal{H})$ satisfying

- $E(\emptyset) = 0$ and $E(X) = 1$,
- For each $U \in \Sigma$, $E(U)$ is a *projection*, that is, $E(U)^* = E(U)^2 = E(U)$,
- $E(U_1 \cap U_2) = E(U_1)E(U_2)$ for all $U_1, U_2 \in \Sigma$, and
- Whenever $\{U_i\}_{i=1}^{\infty} \subset \Sigma$ is a sequence of pairwise disjoint subsets, then

$$E\left(\bigcup_{i=1}^{\infty} U_i\right) = \sum_{i=1}^{\infty} E(U_i).$$

**Lemma 1.2.3.4.** *If $E$ is a spectral measure on $(X, \Sigma, \mathcal{H})$, then for all $g, h \in \mathcal{H}$,*

$$E_{g,h}(U) := \langle E(U)g, h \rangle$$

*defines a complex-valued measure on $(X, \Sigma)$ with total variation at most $\|g\|\|h\|$.*

For a spectral measure $E$ on $(X, \Sigma, \mathcal{H})$ and $f : X \to \mathbb{C}$ a bounded, $\Sigma$-measurable function, we can uniquely define the (operator-valued) integral $\int f dE$ first for simple functions:

$$\phi = \sum_{j=1}^{k} \alpha_j \chi_{U_j} \quad \to \quad \int \phi dE := \sum_{j=1}^{k} \alpha_j E(U_j)$$

Then, for $f$ bounded, measurable, we define $\int f dE$ as the $\mathcal{B}(\mathcal{H})$-norm limit of integrals over simple functions which converge uniformly to $f$ (see Proposition IX.1.10 from [12]).

**Definition 1.2.3.5.** For $x \in \mathcal{B}(\mathcal{H})$, its *spectrum* is

$$\sigma(x) = \{z \in \mathbb{C} | (x - zI) \text{ is not invertible}\}.$$

The spectrum is always non-empty and compact.

An element $x$ of a C* or von Neumann algebra is *normal* if $x^*x = xx^*$.

**Theorem 1.2.3.6.** *(**The Spectral Theorem for C*-Algebras**, IX.2.2 from [12]) If $x \in \mathcal{B}(\mathcal{H})$ is normal, then there is a unique spectral measure $E$ on the Borel subsets of $\sigma(x)$ such that*

- $x = \int z dE(z)$,
- *If $U \subset \sigma(x)$ is relatively open and non-empty, then $E(U) \neq 0$,*
- *For any $a \in \mathcal{B}(\mathcal{H})$, then $ax = xa$ and $a^*x = xa^*$ if and only if $aE(U) = E(U)a$ for every Borel $U \subset \sigma(x)$.*

*Moreover, the continuous functions on the spectrum of $x$ is*

$$C(\sigma(x)) \simeq C^*(x) \text{ (isometric C*-isomorphic)},$$

*the C*-algebra generated by $x$, via the map $\pi_x : f(z) \mapsto \int f(z)dE(z)$. This map is called the continuous functional calculus.*

By taking their respective WOT closures, a version for von Neumann algebras follows (at least, for $\mathcal{H}$ separable).

**Theorem 1.2.3.7.** *(**The Spectral Theorem for von Neumann Algebras**, IX.8.10 from [12]) If $x \in \mathcal{B}(\mathcal{H})$ is normal on $\mathcal{H}$ a separable Hilbert space, then there is a unique spectral measure $E$ on the Borel subsets of $\sigma(x)$ satisfying the above properties. Moreover, the von Neumann algebra generated by $x$ is*

$$W^*(x) \simeq L^\infty(\sigma(x), \mu) \text{ (isometric W*-isomorphic)},$$

*where $\mu$ is a positive, Borel measure on the spectrum of $x$ satisfying*

$$\mu(U) = 0 \textit{ if and only if } E(U) = 0.$$

*Similar to the C*-algebra case, the map is $\pi_x : f(z) \mapsto \int f(z)dE(z)$, but extended WOT-continuously. In this case, it is called the Borel functional calculus.*

**Definition 1.2.3.8.** For a faithful normal state $\tau$ on $\mathcal{B}(\mathcal{H})$ and a normal operator $x \in \mathcal{B}(\mathcal{H})$, $E_\tau(U) := \tau[\pi_x(\chi_U)]$ defines a Borel measure $\mu_x$, called the *spectral measure of $x$ with respect to $\tau$* or sometimes called the *probability density of $x$ with respect to $\tau$*.

**Corollary 1.2.3.9.** *For an element $x$ of a C* or W*-probability space, this measure is the unique compactly supported Borel probability measure on $\mathbb{C}$ whose moments share those of $x$ (moments of a measure are defined in the sense of classical probability: $\mathbb{E}[X^n] = \int x^n d\mu_X$).*

All claims follow directly from the Spectral Theorem.

*1.2.3.5 Free Group Factors*

For this subsection, we will be in the setting of Example 1.2.1.5. Each element of a discrete group $G$ acts on $\ell^2[G]$ via the map $\lambda : G \to \mathcal{U}(\ell^2(G))$ (*unitary* linear operators, that is, those satisfying $a^*a = aa^* = 1$) given by

$$\lambda(g)f(h) = f(g^{-1}h).$$

Extend $\lambda$ linearly to see that all of $\mathbb{C}[G]$ acts on $\ell^2(G)$ this way. Moreover, $\lambda$ is a *-representation of $\mathbb{C}[G]$ into $\mathcal{B}(\ell^2(G))$. Define the *group Von Neumann algebra* for $G$ to be

$$\mathcal{L}(G) := \overline{\lambda(\mathbb{C}[G])}^{WOT} \subset \mathcal{B}(\ell^2(G)).$$

For a von Neumann algebra $\mathcal{M}$, its *center* is $\mathcal{Z}(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}' \subset \mathcal{B}(\mathcal{H})$. Then $\mathcal{M}$ is a *factor* if $\mathcal{Z}(\mathcal{M}) = \mathbb{C}1$.

**Proposition 1.2.3.10.** *$\mathcal{L}(G)$ is a factor if and only if $G$ is ICC, that is, the conjugacy class of every $g \in G$ ($g \neq e$) has infinite cardinality.*

*Proof.* ($\Rightarrow$) (*The easy direction*) Suppose not, that is, there exists $g \in G \backslash \{e\}$ with a finite conjugacy class, say $\{g, h_1^{-1} g h_1, h_2^{-1} g h_2, ..., h_n^{-1} g h_n\}$. Then for every $k \in G$ and every $i \in \{0, ..., n\}$, there exists $j \in \{0, ..., n\}$ such that

$$h_i^{-1} g h_i k = k h_j^{-1} g h_j,$$

where for convenience, we let $h_0 := e$. To see this, apply $k^{-1}$ to the left of $h_i^{-1} g h_i k$. The resulting element must be in the conjugacy class and moreover equal one of the elements listed.

Thus, the operator $\lambda\left(\sum_{i=0}^{n} h_i^{-1} g h_i\right) \in \mathcal{L}(G)$ commutes with all of $\mathcal{L}(G)$. Hence, $\mathcal{L}(G)$ is not a factor.

($\Leftarrow$) (*The harder direction*) Assume $G$ is ICC. This direction has four major steps:

**1)** Let $x \in \mathcal{L}(G)$ and $g \in G$. I aim to show that $\langle x \delta_{g^{-1}h}, \delta_h \rangle$ is constant with respect to $h \in G$.

Take a net $x_i = \lambda\left(\sum_{n=0}^{N_i} \alpha_{i,n} k_{i,n}\right)$ (for $i \in I$ a directed set) that converges to $x$ in the weak operator topology. Then

$$\langle x_i \delta_{g^{-1}e}, \delta_e \rangle = \left\langle \lambda\left(\sum_{n=0}^{N_i} \alpha_{i,n} k_{i,n}\right) \delta_{g^{-1}}, \delta_e \right\rangle = \left\langle \sum_{n=0}^{N_i} \alpha_{i,n} \delta_{k_{i,n} g^{-1}}, \delta_e \right\rangle = \alpha_{i,n_g} \quad \text{(where } k_{n_g} = g).$$

A similar calculation shows $\langle x_i \delta_{g^{-1}h}, \delta_h \rangle = \alpha_{i,n_g}$ as well. Then we have equality in the limit, since $x_i \to x$ in the WOT.

**2)** Let $c_g(x) :=$ the value of that constant map for $g \in G$ and $x \in \mathcal{L}(G)$. Then

$$c_g(x) = \langle x \delta_{g^{-1}}, \delta_e \rangle = \langle x \delta_e, \delta_g \rangle,$$

from which it follows that $x \delta_e = \sum_{g \in G} c_g(x) \delta_g$, since $\{\delta_g\}_{g \in G}$ is an orthonormal basis for $L^2(G)$. Moreover, $\sum_{g \in G} |c_g(x)|^2 < \infty$.

**3)** From this point on, assume $x \in Z(\mathcal{L}(G))$. Let $g, h \in G$. Then

$$\begin{aligned} c_{h^{-1}gh}(x) = \langle x\delta_{h^{-1}g^{-1}hk}, \delta_k\rangle &= \langle x\lambda(h^{-1})\lambda(g^{-1})\lambda(h)\delta_k, \delta_k\rangle \\ &= \langle x\lambda(g^{-1})\lambda(h)\delta_k, \lambda(h)\delta_k\rangle = \langle x\lambda(g^{-1})\delta_{hk}, \lambda(h)\delta_{hk}\rangle \\ &= \langle x\lambda(g^{-1})\delta_k, \lambda(h)\delta_k\rangle = \langle x\delta_{g^{-1}k}, \lambda(h)\delta_k\rangle = c_g(x). \end{aligned}$$

Since $\sum_{g\in G} |c_g(x)|^2 < \infty$ from the previous step, we must have $c_g(x) = 0$ whenever the conjugacy class of $g$ is infinite.

**4)** Since $G$ is ICC, for every $g \in G\backslash\{e\}$ we then have

$$0 = c_g(x) = \langle x\delta_{g^{-1}h}, \delta_h\rangle$$

for every $h \in G$. In other words, $x\delta_{g^{-1}h}$ does not contain a multiple of $\delta_h$ in its basis expansion for any $h \in G$ and $g \in G\backslash\{e\}$. Hence, $x$ must be a multiple of the identity, so $\mathcal{L}(G)$ is a factor. □

Moreover, we have a trace on the group von Neumann algebras: the functional $\tau$, given in Example 1.2.1.5, is called the *von Neumann trace*. Note that it can be written in the form $\tau[x] = \langle x\delta_e, \delta_e\rangle$.

**Lemma 1.2.3.11.** *$\tau$ is a faithful, normal, tracial state on $\mathcal{L}(G)$.*

*Proof.* That $\tau$ is a state is clear. Normality follows from the fact that $\tau$ is a vector state, i.e. $\tau = \langle \cdot\delta_e, \delta_e\rangle$. For traciality, it suffices to check for $\lambda(g), \lambda(h) \in \mathbb{C}[G]$:

$$\tau[\lambda(g)\lambda(h)] = \langle \lambda(gh)\delta_e, \delta_e\rangle = \langle \delta_{gh}, \delta_e\rangle = \begin{cases} 1 & \text{if } gh = e \\ 0 & \text{otherwise.} \end{cases}$$

Since in a group $gh = e$ is equivalent to $hg = e$, we see that $\tau[\lambda(g)\lambda(h)] = \tau[\lambda(h)\lambda(g)]$.

Next, suppose $x \in \mathcal{L}(G)$ satisfies $0 = \tau[x^*x] = \langle x^*x\delta_e, \delta_e\rangle = \langle x\delta_e, x\delta_e\rangle$. Thus, $x\delta_e = 0$. To show that $x = 0$, it suffices to check that $\langle x\delta_g, \delta_h\rangle = 0$ for any $g, h \in G$ (by linearity and continuity). Since $\tau$ is a trace,

$$\langle x\delta_g, \delta_h\rangle = \langle \lambda(h^{-1})x\lambda(g)\delta_e, \delta_e\rangle = \tau[\lambda(h^{-1})x\lambda(g)] = \tau[\lambda(g)\lambda(h^{-1})x] = \langle \lambda(gh^{-1})x\delta_e, \delta_e\rangle = 0.$$

$\square$

In fact, $\tau$ is the only faithful, normal tracial state on $\mathcal{L}(G)$. We call any factor that has only one such functional on it a *type $II_1$-factor*. The rest of Murray and von Neumann's classification system has been omitted for brevity.

Next, we will consider the case $G = \mathbb{F}_n$, the *free group* with $n \in \mathbb{N} \cup \{\infty\}$ generators, which is defined as follows. Start with symbols $\{a_1, ..., a_n\}$, along with their prescribed *inverse* symbols $\{a_1^{-1}, ..., a_n^{-1}\}$ and a prescribed *identity* symbol $e$. Then $\mathbb{F}_n$ is defined to be the set of all words of arbitrary length, consisting of any of the aforementioned symbols, that have been reduced using the following relations:

$$\begin{aligned} &a_i a_i^{-1} = a_i^{-1} a_i = e \\ &a_i e = e a_i = a_i \\ &a_i^{-1} e = e a_i^{-1} = a_i^{-1}. \end{aligned}$$

The binary operation of concatenation immediately followed by reduction using the above relations makes $\mathbb{F}_n$ a group. For $n \geq 2$, it is ICC, so $\mathcal{L}(\mathbb{F}_n)$ is a factor, called a *free group factor*.

It is known that $\mathbb{F}_n$ is (group) isomorphic to $\mathbb{F}_m$ if and only if $n = m$. However, the corresponding question for $\mathcal{L}(\mathbb{F}_n)$, the *Free Group Factor Isomorphism Problem*, remains unanswered. The concept of free independence was introduced by Dan Virgil Voiculescu in the 1980s to, among other things, gain a foothold into solving this mystery.

### 1.2.4 Free Probability and von Neumann Algebras

Voiculescu's approach soon bore promising fruit. For instance, the following theorem shows that, in principle, all information about a von Neumann algebra is contained in the joint distribution of a generating set with respect to a faithful normal state. Moreover, this gives a plan of attack: show that generating subsets of two von Neumann algebras have the same joint distributions.

**Theorem 1.2.4.1.** *Let $\mathcal{M} = vN\{a_1, ..., a_n\}$ be a von Neumann algebra generated by elements $a_1, ..., a_n$, with faithful normal state $\phi : \mathcal{M} \to \mathbb{C}$, and $\mathcal{N} = vN\{b_1, ..., b_n\}$ be a von Neumann algebra generated by elements $b_1, ..., b_n$, with faithful normal state $\psi : \mathcal{N} \to \mathbb{C}$. If $\{a_1, ..., a_n, a_1^*, ..., a_n^*\}$ and $\{b_1, ..., b_n, b_1^*, ..., b_n^*\}$ have the same joint distributions with respect to $\phi$ and $\psi$, respectively, then the map $a_i \mapsto b_i$ extends to a W*-isomorphism of $\mathcal{M}$ and $\mathcal{N}$.*

The proof is simply observing that the GNS constructions of $\mathcal{M}$ with respect to $\phi$ and $\mathcal{N}$ with respect to $\psi$ are isomorphic.

Returning to the free group factors, the free group is generated by elements $a_1, ..., a_n$, so $\lambda(\mathbb{C}[\mathbb{F}_n])$ is generated by $\lambda(a_1), ..., \lambda(a_n)$. Note that for each $i$, $\lambda(x_i)$ is a *Haar unitary* with respect to $\tau$, that is, each $\lambda(x_i)$ is a unitary operator such that $\tau[\lambda(x_i)^n] = \delta_{n=0}$ for $n \in \mathbb{Z}$.

**Proposition 1.2.4.2.** *In the non-commutative *-probability space $(\lambda(\mathbb{C}[\mathbb{F}_n]), \tau)$, the generators $\lambda(a_1)$, ..., $\lambda(a_n)$ are freely independent.*

*Proof.* Take polynomials $P_i(\lambda(x_{u(i)}), \lambda(x_{u(i)}^{-1})) = \sum_{k\in\mathbb{Z}} \alpha_k^i \lambda(x_{u(i)}^k)$ such that $\tau[P_i(\lambda(x_{u(i)}), \lambda(x_{u(i)}^{-1}))] = 0$ for all $i$ and $u(1) \neq u(2) \neq ... \neq u(m)$.

First note that $\alpha_0^{(i)} = 0$ for each $P_i$ by assumption. Also

$$\prod_{i=1}^{m} P_i(\lambda(x_{u(i)}), \lambda(x_{u(i)}^{-1})) = \sum_{\vec{k}\in\mathbb{Z}^m} \alpha_{k(1)}^1 ... \alpha_{k(m)}^m \lambda(x_{u(1)}^{k(1)} ... x_{u(m)}^{k(m)}),$$

where all $k(j) \neq 0$. By the alternating property, the word $x_{u(1)}^{k(1)} ... x_{u(m)}^{k(m)}$ never equals $e$. Hence, $\tau$ applied to each term of the sum is zero. □

Therefore, by Theorem 1.2.4.1, a von Neumann algebra is isomorphic to the free group factor $\mathcal{L}(\mathbb{F}_n)$ if we can find $n$ Haar unitaries that are free with respect to a faithful normal state and generate the von Neumann algebra. However, the next theorem shows that we can lower our standards a bit. Thanks to the Borel functional calculus given to us via the Spectral Theorem, we can reshape the generators (to an extent) into generators with a wide variety of distributions.

**Theorem 1.2.4.3.** *Let $\mathcal{M} = vN\{x_1, ..., x_n\}$ be a von Neumann algebra generated by elements $x_1, ..., x_n$, with faithful normal state $\tau : \mathcal{M} \to \mathbb{C}$. Assume that*

- *$x_1, ..., x_n$ are freely independent with respect to $\tau$, and*
- *each $x_i$ is normal and with non-atomic spectral measure with respect to $\tau$.*

*Then $\mathcal{M} \simeq \mathcal{L}(\mathbb{F}_n)$.*

Throughout the 1990s, other results of a similar spirit (by Voiculescu, Dykema, among others) have been proven.

### 1.2.5 Free Products

Here, I will begin by describing in more detail the free product of groups $G_1, ..., G_n$ with respective identity elements $e_i \in G_i$. Any word $g_1...g_k$ can be reduced by identifying $xe_j = e_jx = x$ for $x \in G_i$ and $e_j \in G_j$ for any $i, j$, and by identifying $wxyz = w(xy)z$ if $x, y \in G_i$ for the same $i$. Through these, any word that is not just $e_i$ has a unique reduced form:

$$g_1...g_k, \text{ where } g_i \in G_{u(i)} \backslash \{e_{u(i)}\}, \ u(1) \neq u(2) \neq ... \neq u(k).$$

Define the *free product* $*_{i=1}^n G_i$ of these groups to be the set of such reduced words in elements of the $G_i$, with identity $e$, inverse

$$(g_1...g_k)^{-1} = g_k^{-1}...g_1^{-1},$$

and group binary operation (product)

$$(g_1...g_k)(h_1...h_\ell) = \text{ reduced version of } g_1...g_kh_1...h_\ell.$$

For $(\mathcal{A}_1, \phi_1), ..., (\mathcal{A}_n, \phi_n)$ noncommutative probability spaces, the following serves as an analogue of the above construction. Denote $\mathcal{A}_i^\circ = \{a \in \mathcal{A}_i \mid \phi_i[a] = 0\}$. Similarly, for $a$ in any NCPS $(\mathcal{A}, \phi)$, we write $a^\circ = a - \phi[a]$ (its *centered*, or mean-zero, counterpart). Let

$$\mathcal{A} = \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{u(1)\neq ...\neq u(n)} \mathcal{A}_{u(1)}^\circ \otimes ... \otimes \mathcal{A}_{u(n)}^\circ.$$

By abuse of notation, we will write $a_1 a_2 ... a_n := a_1 \otimes a_2 \otimes ... \otimes a_n$. Define

$$(a_1 ... a_k)^* = a_k^* ... a_1^*,$$

and multiplication, for $a_i \in \mathcal{A}_{u(i)}^\circ$, $b_i \in \mathcal{A}_{v(i)}^\circ$ where $u(1) \neq ... \neq u(k)$ and $v(1) \neq ... \neq v(\ell)$, by

$$(a_k ... a_1)(b_1 ... b_\ell) = a_k ... a_2 (a_1 b_1)^\circ b_2 ... b_\ell + \phi_{u(1)}[a_1 b_1] a_k ... a_3 (a_2 b_2)^\circ b_3 ... b_\ell + ...$$
$$+ \phi_{u(1)}[a_1 b_1] ... \phi_{u(j-1)}[a_{j-1} b_{j-1}] a_k ... a_j b_j ... b_\ell,$$

where $j \leq \min\{k, \ell\}$ is the first index such that $u(j) \neq v(j)$ (if it exists).

On this $*$-algebra $\mathcal{A}$, define the *free product state* $\phi := *_{i=1}^n \phi_i$ by $\phi[1] = 1$ and $\phi[a_1 a_2 ... a_n] = 0$ for all $a_i \in \mathcal{A}_{u(i)}$ such that $u(1) \neq ... \neq u(k)$ and $\phi_{u(i)}[a_i] = 0$. The noncommutative probability space $(\mathcal{A}, \phi)$ is called the *reduced free product*. In other words, this is a NCPS $(\mathcal{A}, \phi)$ such that each $(\mathcal{A}_i, \phi_i)$ embeds into $\mathcal{A}$ as subalgebras in a way that the $\mathcal{A}_i$ are freely independent with respect to $\phi$, and moreover, $\phi|_{\mathcal{A}_i} = \phi_i$.

#### *1.2.5.1 Free Dimension*

Here, we will use notation from [13]. For von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with respective traces $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, let $\underset{\alpha}{\mathcal{A}} \oplus \underset{\beta}{\mathcal{B}}$, for $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, denote the algebra $\mathcal{A} \oplus \mathcal{B}$ with trace $\tau(a, b) = \alpha \tau_{\mathcal{A}}(a) + \beta \tau_{\mathcal{B}}(b)$. This notation naturally extends to direct sum algebras with more summands.

In 1992, in the course of investigations of the free group factor isomorphism problem, Ken

Dykema defined the *free dimension* of *-algebras of the form

$$\mathcal{A} = \underset{\gamma_0}{\mathcal{L}(\mathbb{F}_r)} \oplus \underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2}{\mathbb{C}} \oplus ... \oplus \underset{\gamma_n}{\mathbb{C}}$$

to be

$$\text{fdim}(\mathcal{A}) := r\gamma_0^2 + \sum_{\substack{0 \le i,j \le n \\ i \ne j}} \gamma_i \gamma_j.$$

This includes the case $\text{fdim}(\mathcal{L}(\mathbb{F}_r)) = r$.

This is used to describe the free products of such algebras in the following way:

**Proposition 1.2.5.1.** *(Proposition 2.4 from [13]) Let*

$$\mathcal{A} = \underset{\alpha_0}{\mathcal{L}(\mathbb{F}_r)} \oplus \underset{\alpha_1}{\mathbb{C}} \oplus \underset{\alpha_2}{\mathbb{C}} \oplus ... \oplus \underset{\alpha_n}{\mathbb{C}} \quad (n \ge 0, r \ge 1, \alpha_0 \ge 0)$$

$$\mathcal{B} = \underset{\beta_0}{\mathcal{L}(\mathbb{F}_s)} \oplus \underset{\beta_1}{\mathbb{C}} \oplus \underset{\beta_2}{\mathbb{C}} \oplus ... \oplus \underset{\beta_m}{\mathbb{C}} \quad (m \ge 0, s \ge 1, \beta_0 \ge 0),$$

*where* $\alpha_0 + \beta_0 > 0$*. Then their free product*

$$\mathcal{A} * \mathcal{B} \simeq \mathcal{L}(\mathbb{F}_t) \bigoplus_{\substack{1 \le i \le n \\ 1 \le j \le m}} \underset{\gamma_{ij}}{\mathbb{C}},$$

*where* $\gamma_{ij} = \max\{\alpha_i + \beta_j - 1, 0\}$*, and* $t$ *is chosen so that fdim*$(\mathcal{A} * \mathcal{B}) =$ *fdim*$(\mathcal{A}) +$ *fdim*$(\mathcal{B})$*.*

### 1.2.6 Combinatorics of Partitions and Free Cumulants

A *partition* $\pi$ of a subset $S \subset \mathbb{N}$ is a collection of disjoint subsets of $S$ (called $blocks$ of $\pi$) whose union equals $S$. We will use $i \overset{\pi}{\sim} j$ to say that $i$ and $j$ are in the same block of $\pi$. In this paper, we will only be concerned with partitions of $[n] := \{1, 2, ..., n\}$.

Let NC$(n)$ denote the *noncrossing partitions* over $[n]$, that is, those partitions $\pi$ such that there are no $i < j < k < \ell$ such that $i \overset{\pi}{\sim} k$ and $j \overset{\pi}{\sim} \ell$ unless all four are in the same block. If $i \in [n]$ is the first element of its block, we will call it an *opening element*, while the last of its block will be called a *closing element*. If $i$ is neither, it will be called a *middle element*.

For distinct blocks $V$ and $W$ of a noncrossing partition $\pi$, V is said to be *inner with respect to* $W$ if $o_W < o_V < c_V < c_W$, where $o_V$ and $o_W$ are the opening elements of $V$ and $W$, respectively, while $c_V$ and $c_W$ are their closing elements. We will simply say a block is *inner* if it is inner with respect to some block, and *outer* if it is not.

Finally, let $\mathrm{Int}(n)$ denote the *interval partitions* over $[n]$, that is, those partitions $\pi$ such that whenever $i < j$ and $i \overset{\pi}{\sim} j$, we have $i \overset{\pi}{\sim} k$ for all $i < k < j$.

The *free cumulants* $R[X_{j(1)}, ..., X_{j(k)}] \in \mathbb{C}$ of $X_1, ..., X_n \in (\mathcal{A}, \psi)$ (*with respect to* $\psi$) are defined inductively via the *moment-cumulant formula*

$$\psi[X_{i(1)}...X_{i(k)}] = \sum_{\pi \in \mathrm{NC}(n)} R_\pi[X_{i(1)}, ..., X_{i(k)}],$$

where

$$R_\pi[X_1, ..., X_k] = \prod_{V \in \pi} R[X_{V(1)}, ..., X_{V(|V|)}].$$

Compare this to the classical moment-cumulant formula (which may, for the purposes of this section, serve as a definition of classical cumulants $K[X_{j(1)}, ..., X_{j(k)}] \in \mathbb{C}$):

$$\mathbb{E}[X_{i(1)}...X_{i(k)}] = \sum_{\pi \in \mathrm{P}(n)} K_\pi[X_{i(1)}, ..., X_{i(k)}],$$

where $\mathrm{P}(n)$ denotes the set of all partitions of $[n]$ and

$$K_\pi[X_1, ..., X_k] = \prod_{V \in \pi} K[X_{V(1)}, ..., X_{V(|V|)}].$$

The appearance of non-crossing partitions in the formula makes sense in light of the following proposition. There is a similar result for the classical setting as well.

**Proposition 1.2.6.1.** *(**Speicher**) Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_k \subset \mathcal{A}$ are free if and only if their mixed free cumulants (that is, free cumulants of elements of more than one $A_i$) are zero.*

To prove this, we need a better understanding of the free cumulants.

**Theorem 1.2.6.2.** *Let $n_1, ..., n_r \in \mathbb{Z}_+$ such that $n = n_1 + ... + n_r$. Take $a_1, ..., a_n \in (\mathcal{A}, \phi)$ a NCPS. Let*

$$A_1 = a_1...a_{n_1}, A_2 = a_{n_1+1}...a_{n_1+n_2}, ..., A_r = a_{n_1+...+n_{r-1}+1}...a_n.$$

*Then*

$$R_r[A_1, ..., A_r] = \sum_{\substack{\pi \in NC(n) \\ \pi \vee \sigma = \hat{1}_n}} R_\pi[a_1, ..., a_n],$$

*where $\sigma$ is the partition whose blocks are $(1, ..., n_1), (n_1 + 1, ..., n_1 + n_2), ..., (n_1 + ... + n_{r-1} + 1, ..., a_n)$, and $\vee$ denotes the join, whose blocks are defined by the equivalence relation on $[n]$ generated by those of $\pi$ and $\sigma$.*

*Proof.* (of Proposition 1.2.6.1)

($\Leftarrow$) For $n \geq 2$, take $a_i \in \mathcal{A}_{j(i)}$ such that $j(1) \neq j(2) \neq ... \neq j(m)$ and $\phi[a_i] = 0$. We need to show that $\phi[a_1 a_2 ... a_n] = 0$. Then

$$\phi[a_1 a_2 ... a_n] = \sum_{\pi \in \mathrm{NC}(n)} R_\phi[a_1, ..., a_n] = \sum_{\pi \in \mathrm{NC}_{ns}(n)} R_\phi[a_1, ..., a_n],$$

where the sum is taken over all non-crossing partitions with no singletons since $R_\phi[a_i] = \phi[a_1] = 0$. Each of the remaining partitions has a block containing a pair of consecutive numbers, so the cumulant term corresponding to the block is zero. Hence, the entire sum is zero.

($\Rightarrow$)

For this, we need to use the following lemma: Every cumulant $R[a_1, ..., a_n] = 0$ that has $a_i = 1$ for some $i$. By multilinearity of $R$ and the lemma, without loss of generality, we can take $a_i \in \mathcal{A}_{u(i)}$ such that $\phi[a_i] = 0$ for all $i$.

For $n = 2$, the mixed assumption implies $u(1) \neq u(2)$, so by freeness,

$$R[a_1, a_2] = \phi[a_1 a_2] = \phi[a_1]\phi[a_2].$$

Next, group each $u(i)$ so that

$$u(1) = ... = u(v(1))$$
$$u(v(1)+1) = ... = u(v(2))$$
$$u(v(j-1)+1) = ... = u(v(j)),$$

where $u(v(1)) \neq u(v(2)) \neq ... \neq u(v(n))$. Then denote

$$A_1 = a_1 ... a_{v(1)}, A_2 = a_{v(1)+1} ... a_{v(2)}, ..., A_j = a_{v(j-1)+1} ... a_{v(j)}.$$

By the induction hypothesis, $R[A_1, A_2, ..., A_j] = 0$ since $j < n$. But by Theorem 1.2.6.2 this also equals

$$\sum_{\pi \vee \sigma = \hat{1}_n} R_\pi[a_1, ..., a_n]$$

for some $\sigma \in \text{Int}(n)$, $\sigma \neq \hat{1}_n$. Take $\pi \neq \hat{1}_n$. By the induction hypothesis, if $\pi$ contributes a nonzero term to the sum, then all $a_i$ in the same block of $\pi$ must be in the same subalgebra. Also, by construction, the same applies to $\sigma$. So $\pi \vee \sigma = \hat{1}_n$, so all $a_1, ..., a_n$ must be in the same subalgebra, a contradiction. Hence, all terms corresponding to $\pi \neq \hat{1}_n$ are zero, so $R_n[a_1, ..., a_n] = 0$ as well. □

The analogous *Boolean cumulants* $B[X_{j(1)}, ..., X_{j(k)}] \in \mathbb{C}$ are defined in almost exactly the same manner as the free case:

$$\psi[X_{i(1)}, ..., X_{i(k)}] = \sum_{\pi \in \text{Int}(n)} B_\pi[X_{i(1)}, ..., X_{i(k)}],$$

where

$$B_\pi[X_1, ..., X_k] = \prod_{V \in \pi} B[X_{V(1)}, ..., X_{V(|V|)}].$$

As one can expect, Boolean independence is equivalent to the vanishing of mixed Boolean cumulants.

Moreover, Belinschi and Nica [14] showed the following relation between Boolean and free cumulants:

**Theorem 1.2.6.3.**

$$B[X_1, ..., X_n] = \sum_{\pi \in \widetilde{\mathrm{NC}}(n)} R_\pi[X_1, ..., X_n]$$

*where* $\widetilde{\mathrm{NC}}(n) = \{\pi \in \mathrm{NC}(n) | 1 \overset{\pi}{\sim} n\}$ *(called the set of irreducible non-crossing partitions).*

To see this, observe that the restriction of $\pi \in$ NC$(n)$ to each block of the smallest $\sigma \in$ Int$(n)$ such that $\pi \leq \sigma$ connects the minimum and maximum of that block.

*1.2.6.1 New Partial Order on NC*$(n)$

Let $\ll$ denote a partial order on NC$(n)$ defined by

$$\sigma \ll \pi \quad \Leftrightarrow \quad \sigma \leq \pi \text{ and } \forall\, V \in \pi,\ \min(V) \overset{\sigma}{\sim} \max(V),$$

where $\leq$ denotes the usual partial order (*reverse refinement*) on the set of all partitions of $[n]$:

$$\sigma \leq \pi \quad \Leftrightarrow \quad \forall\, B \in \sigma,\ \exists C \in \pi \text{ such that } B \subseteq C.$$

Remark 2.14 in the same work [14] gives the following:

**Lemma 1.2.6.4.** *For* $\sigma \in$ *NC*$(n)$*,*

$$\{\pi \in \mathrm{NC}(n) : \sigma \ll \pi\} \simeq \{S \subset \sigma : S \textit{ contains the outer blocks of } \sigma\} =: S_\sigma.$$

*The bijection is given by*

$$\pi = \{B_1, ..., B_k\} \mapsto \{o(\sigma|_{B_1}), o(\sigma|_{B_2}), ..., o(\sigma|_{B_k})\},$$

*where* $o(\sigma|_{B_i})$ *denotes the unique (by the partial order) outer block of the partition* $\sigma|_{V_i}$*.*

*1.2.6.2 The Free Central Limit Theorem*

In classical probability, the Gaussian (normal) distribution plays many important roles. Its counterpart in free probability is Wigner's *semi-circular distribution with variance* $t > 0$, defined via the density

$$d\mu(x) = \frac{1}{2\pi t}\sqrt{4t - x^2}\mathbf{1}_{[-2\sqrt{t},2\sqrt{t}]}dx.$$

In free probability, moments and free cumulants uniquely determine a distribution, so those numbers, respectively, for semi-circular distributions are

$$\begin{aligned} m_{2n} &= t^n C_n \\ m_{2n+1} &= 0 \text{ for all } n \\ R_2 &= m_2 = t \\ R_n &= 0 \text{ for } n \neq 2, \end{aligned}$$

where $C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$ is the $n$th *Catalan number*.

The following is not only an analogue of the Central Limit Theorem in free probability, but a natural way to justify giving semi-circular distributions the title of "free analogue of normal variables". Moreover, this side excursion into Speicher's combinatorial proof illustrates his motivation behind the free cumulants.

**Theorem 1.2.6.5.** ***(Free Central Limit Theorem)*** *If $\{X_k\}_{k\in\mathbb{N}} \subset (\mathcal{A}, \phi)$ is a sequence of freely independent, identically distributed random variables with common mean and variance*

$$\phi[X_i] = 0, \quad \phi[X_i^2] = t, \text{ respectively, then}$$

$$S_k := \frac{X_1 + ... + X_k}{\sqrt{k}}$$

*converges in distribution to the semi-circular distribution with mean 0 and variance $t$.*

*Proof.* First, we compute the $n$th moment of $S_k$:

$$\phi[S_k^n] = \frac{1}{k^{n/2}} \sum_{r:[n]\to[k]} \phi[x_{r(1)}...x_{r(n)}]. \tag{1.2}$$

For a multi-index $i = (i_1, ..., i_n)$, define its *kernel*, ker$i$, to be the partition of $[n]$ whose blocks are defined by

$$k \text{ and } \ell \text{ are in the same block } \leftrightarrow i_k = i_\ell.$$

Then we have this lemma: If ker$i$ = ker$j$, then $\phi[x_{i(1)}...x_{i(n)}] = \phi[x_{j(1)}...x_{j(n)}]$. This follows from free independence and the assumption of identical distribution of the variables. As a consequence of the lemma, we may define $\phi[\pi]$ to be the common value of $\phi[x_{i(1)}...x_{i(n)}]$ for all $i$ with ker$i = \pi \in \mathcal{P}(n)$. Then (1.2) simplifies to

$$\phi[S_k^n] = \frac{1}{k^{n/2}} \sum_{\pi\in\mathcal{P}(n)} \phi[\pi] \cdot |\{i : [n] \to [k] | \ker i = \pi\}|. \tag{1.3}$$

We have

$$|\{i : [n] \to [k] | \ker i = \pi\}| = k(k-1)...(k - \#(\pi) + 1)$$

(where $\#(\pi)$ is the number of blocks in $\pi$), since we have $k$ choices for the first block, $k-1$ choices for the second, and so on. Thus, 1.3 simplifies to

$$\phi[S_k^n] = \frac{1}{k^{n/2}} \sum_{\pi\in\mathcal{P}(n)} \phi[\pi] k(k-1)...(k - \#(\pi) + 1). \tag{1.4}$$

Note that the number of terms in the sum no longer depends on $k$, so we are in position to take the limit as $k \to \infty$. First, note that

$$k(k-1)...(k - \#(\pi) + 1) \sim k^{\#(\pi)} \text{ as } k \to \infty.$$

Next, if $\pi$ has a singleton block, then we have $\phi[\pi] = 0$, due to freeness and the fact that all $X_i$

have mean zero, since $\phi[\pi]$ will be the product of $\phi[X_i]$ (for some $i$) and other moments.

The remaining terms correspond to partitions with blocks of size at least 2, so they can have at most $n/2$ blocks. Taking the limit of these terms, we get

$$\lim_{k\to\infty} \frac{k^{\#(\pi)}}{k^{n/2}} = \begin{cases} 1, & \text{if } \#(\pi) = n/2, \\ 0, & \text{if } \#(\pi) < n/2. \end{cases}$$

Hence, the only partitions with (potentially) nonzero terms in the sum are the *pairings* of $[n]$. Note that we have also proven that odd moments are zero in the limit. We can rewrite 1.4 as

$$\phi[S_k^n] = \sum_{\pi \in \mathcal{P}_2(n)} \phi[\pi]. \tag{1.5}$$

Due to free independence, $\phi[\pi] = t^n$ if the pairing $\pi$ is non-crossing, and 0 if not. Hence,

$$\lim_{k\to\infty} \phi[S_k^{2n}] = t^n |\mathrm{NC}_2(2n)|.$$

$|\mathrm{NC}_2(2n)|$ is known to be the $n$th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. Hence, $S_k$ converges in distribution to a semi-circular element with mean zero and variance $t$. □

### 1.2.7 Free Additive Convolutions

In classical probability, the distribution of $X+Y$ for $X$ and $Y$ independent is obtained through convolution of their density functions $f_X$ and $f_Y$ (provided they are absolutely continuous). The analogous construction, the *free additive convolution* for $X$ and $Y$ freely independent, is constructed via the corresponding spectral measures $\mu$ and $\nu$ (with respect to state $\phi$) for the distributions of $X$ and $Y$, respectively; this approach also works for measures with atomic components. The first step in computing their free convolution, denoted $\mu \boxplus \nu$, is the following.

**Definition 1.2.7.1.** Given corresponding probability measure $\mu$ to a random variable $X$, its *Cauchy* *transform* is

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t).$$

The integral converges to an analytic function on the upper half-plane $\mathbb{C}^+$, with range contained in the lower half-plane $\mathbb{C}^-$.

The following two propositions allow us to recover the probability measure $\mu$ from its Cauchy transform. These versions of the Stieltjes inversion formula are from [15].

**Theorem 1.2.7.2.** ***(Stieltjes Inversion Formula)*** *If $\mu$ is a probability measure on $\mathbb{R}$ and $G_\mu$ is its Cauchy transform, then for $a < b$ we have:*

$$\mu((a,b)) = \lim_{y \to 0^+} \frac{-1}{\pi} \int_a^b \Im G_\mu(x+iy)dx - \frac{1}{2}\mu(\{a,b\}),$$

*where $x + iy$ is the standard form of a complex number.*

*Moreover, if $\mu$ and $\nu$ are probability measures such that $G_\mu = G_\nu$, then $\mu = \nu$.*

**Proposition 1.2.7.3.** *If $\mu$ is a probability measure on $\mathbb{R}$ and $G_\mu$ is its Cauchy transform, then for all $a \in \mathbb{R}$,*

$$\mu(\{a\}) = \lim_{\measuredangle z \to a} (z-a)G(z),$$

*where we take the non-tangental limit of $(z-a)G_\mu(z)$, that is, for $f : \mathbb{C}^+ \to \mathbb{C}$ and $a \in \mathbb{R}$, $\lim_{\measuredangle z \to a} f(z) = b$ if for every $\theta > 0$, $\lim_{z \to a} f(z) = b$ when $z$ is restricted to the cone $\{x+iy | y > 0$ and $|x-a| < \theta y\} \subset \mathbb{C}^+$.*

**Definition 1.2.7.4.** The corresponding *R-transform* $R_\mu(z)$ of $\mu$ is defined by the relation $G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z$.

Then for $X$ and $Y$ freely independent (with corresponding probability measures $\mu$ and $\nu$, respectively), we have

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z). \tag{1.6}$$

This gives a clear approach for computing the free additive convolutions of two free variables.

First, find their Cauchy transforms, then their R-transforms, add them together, then reverse the process.

Of great interest are the *free additive convolution powers*, that is, measures $\mu^{\boxplus n} := \mu \boxplus ... \boxplus \mu$ ($n$ times). Unlike the classical setting, the convolution powers $\mu^{\boxplus t}$ where $t > 1$ is not necessarily an integer, a construction introduced by Nica and Speicher ([16]), can be defined for any distribution. To begin, let $x \in (\mathcal{A}, \phi)$ (a noncommutative probability space) with distribution $\mu$. Choose a projection $p$ ($p = p^2 = p^*$) that is freely independent from it, with $\phi[p] = \frac{1}{t}$. Then we have the *compressed* noncommutative probability space

$$\left(\mathcal{A}_t, \tilde{\phi}\right) = \left(p\mathcal{A}p, \frac{1}{t}\phi|_{p\mathcal{A}p}\right),$$

and $pxp$ has distribution $\mu^{\boxplus t}$, where $R_{\mu^{\boxplus t}}(z) = tR_\mu(z)$. The multivariable case is done in a similar fashion, with the same relation applying to their joint distribution.

An equivalent form of the R-transform is through the free cumulant generating function:

**Proposition 1.2.7.5.** *Let $R_n^\mu$ denote the $n$th free cumulant of $X$ (with distribution $\mu$). Then $R(z) = \frac{C(z)-1}{z} = \sum_{n=0}^{\infty} R_{n+1}^\mu z^n$, where $C(z)$ is the free cumulant generating function for $X$. So, we can identify the $t$th convolution power of a joint distribution $\mu$ through the relation $R_n^{\mu^{\boxplus t}} = tR_n^\mu$ between free cumulants.*

We conclude this section with the Boolean counterparts of the above.

**Definition 1.2.7.6.** For $X, Y \in (\mathcal{A}, \phi)$ Boolean independent with distributions $\mu$ and $\nu$, respectively, then the distribution of $X + Y$ is denoted $X \uplus Y$, their *Boolean (additive) convolution*.

We can similarly identify the $t$th Boolean convolution power of $\mu$, denoted $\mu^{\uplus t}$, by the relation $B_n^{\mu^{\uplus t}} = tB_n^\mu$ between Boolean cumulants. For brevity, we take this relation as the definition.

**Lemma 1.2.7.7.** *Let $\mu$ be a probability measure on $\mathbb{R}$. Then $\mu$ and $\mu^{\uplus t}$ have the same support for their absolutely continuous parts.*

*Proof.* Denote $G(z) = G_\mu(z)$ and $G_t(z) = G_{\mu^{\uplus t}}(z)$. Then

$$\frac{1}{G_t(z)} - z = t\left(\frac{1}{G(z)} - z\right).$$

Thus

$$G_t(z) = \frac{1}{z + t\left(\frac{1}{G(z)} - z\right)} = \frac{G(z)}{(1-t)zG(z) + t}. \tag{1.7}$$

Therefore

$$\Im G_t(z) = \frac{\Im(G(z)((1-t)\bar{z}\bar{G(z)} + t))}{|(1-t)zG(z) + t|^2}.$$

Now taking $z$ real, we get

$$\begin{aligned}\Im G_t(x) &= \frac{\Im G(x)((1-t)x\Re G(x) + t) + \Re G(x)(1-t)x\Im G(x)}{|(1-t)xG(x) + t|^2} \\ &= \frac{2(1-t)x\Re G(x) + t}{|(1-t)xG(x) + t|^2}\Im G(x)\end{aligned}$$

So except at the special points where $(1-t)xG(x) + t = 0$, if $\Im G(x) = 0$ then $\Im G_t(x) = 0$. By replacing $t$ with $1/t$, we obtain the opposite relation. □

### 1.2.8 Conditionally Free Independence

In 1991, Marek Bozejko and Roland Speicher generalized these ideas to *conditionally free independence* ([17]), whose construction depends on a relationship between the two states which resembles absolute continuity from measure theory. Several von Neumann algebras have been constructed as the similarly defined *conditionally free product*, and for some, there exist clear relationships with their usual free products.

**Definition 1.2.8.1.** To begin, let $(\mathcal{A}, \phi)$ be a noncommutative probability space, with a second vector state $\psi$. Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n \subset \mathcal{A}$ are *conditionally freely independent* with respect to $\phi$

and $\psi$ if, whenever $u_i \in \mathcal{A}_{j(i)}$ for $j(1) \neq j(2) \neq ... \neq j(m)$ with $\psi[u_i] = 0$ for all $i$, we have

$$\phi[u_1 u_2 ... u_m] = \phi[u_1]\phi[u_2]...\phi[u_m], \text{ and}$$
$$\psi[u_1 u_2 ... u_m] = 0.$$

In other words, the $\phi$-mean of an alternating product of $\psi$-mean zero variables is just the product of the variables' respective $\phi$-means, in addition to free independence with respect to $\psi$.

Compare this to the usual notion of free independence. This definition is also reminiscent of absolute continuity of measures ($\mu$ is absolutely continuous with respect to $\nu$ if for every measurable set $A$, $\nu(A) = 0$ implies $\mu(A) = 0$). Moreover, for $(\mathcal{B}, \phi)$ a *non-unital* *-probability space, $\mathcal{A} = \mathbb{C} \oplus \mathcal{B}$ its unitization, and $\tilde{\phi}$ the canonical extension of $\phi$ to $\mathcal{A}$, and the state $\psi$ on $\mathcal{A}$ by $\psi[\lambda 1 + b] = \lambda$, we have all elements of $\mathcal{A}$ are free with respect to $\psi$, and $\{\lambda_1 + b_1, \ldots, \lambda_n + b_n\}$ are conditionally free with respect to $(\tilde{\phi}, \psi)$ if and only if $\{b_1, \ldots, b_n\} \subset \mathcal{B}$ are Boolean independent in $(\mathcal{B}, \phi)$.

**Lemma 1.2.8.2.** *Let $(\mathcal{A}, \Phi, \Psi)$ be a unital two-state probability space, and $\{\mathcal{A}_i\}$ non-unital *-subalgebras.*

*a. If the $\mathcal{A}_i$'s are free with respect to $\psi$, then so are the subalgebras generated by $\mathcal{A}_i$ and $1$.*

*b. If the $\mathcal{A}_i$'s are conditionally free with respect to $(\phi, \psi)$, then so are the subalgebras generated by $\mathcal{A}_i$ and $1$.*

*Proof.* Part (a) follows from Proposition 1.2.6.1, as well as the lemma used in its proof (all subalgebras are free from $\mathbb{C}1$). For part (b), it follows directly from the definition that scalars are conditionally free from any other elements. So $R^{\Phi_t, \Psi_t}[a_1, \ldots, a_n] = 0$ if any of the $a_i$ is a scalar. The result follows. $\square$

The *conditionally free product* is constructed in a similar spirit to conditional freeness, and in a similar fashion to the usual free product (see Subsection 1.2.5):

**Definition 1.2.8.3.** For $(\mathcal{A}_1, \phi_1, \psi_n)$, ..., $(\mathcal{A}_n, \phi_n, \psi_n)$ noncommutative probability spaces with two states each, denote $\mathcal{A}_i^\circ = \{a \in \mathcal{A}_i \,|\, \psi_i[a] = 0\}$, and let

$$\mathcal{A} = \mathbb{C}1 \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{u(1) \neq \ldots \neq u(n)} \mathcal{A}_{u(1)}^\circ \otimes \ldots \otimes \mathcal{A}_{u(n)}^\circ.$$

On this $*$-algebra $\mathcal{A}$, define the *conditionally free product state* $\phi := *_{i=1}^{n}(\phi_i, \psi_i)$ by $\phi[1] = 1$ and $\phi[a_1 a_2 \ldots a_n] = \phi_{i(1)}[a_1] \ldots \phi_{i(n)}[a_n]$ for all $a_i \in \mathcal{A}_{u(i)}$ such that $u(1) \neq \ldots \neq u(k)$ and $\psi_{u(i)}[a_i] = 0$. Define a second state $\psi$ as the usual free product state of the $\psi_i$. The noncommutative probability space $(\mathcal{A}, \phi, \psi)$ is called the *conditionally free product*. Note that if $\phi_i = \psi_i$ for all $i$, we get the usual free product.

In [17], the authors describe the corresponding additive convolution and examine it from both a combinatorial point of view (conditionally free cumulants) and an analytic point of view (the R-transform).

**Definition 1.2.8.4.** The *conditionally free convolution* of $X$ and $Y$ conditionally free is defined in a similar way to the usual free convolution: as the $\phi$-distribution of $X + Y$, where $\phi$ is the conditionally free product state. To be more precise, let $\mu_1, \mu_2, \nu_1$, and $\nu_2$ be compactly supported measures on $\mathbb{R}$, which can be identified with a state on the $*$-algebra $\mathbb{C}\langle X\rangle$, via, for instance

$$\mu_1(X^n) = \int t^n d\mu_1(t).$$

Assume $\mu_1$ is the distribution of $X$, $\mu_2$ is the distribution of $Y$, and let $\phi = (\mu_1, \nu_1) * (\mu_2, \nu_2)$ be their conditionally free product state on $\mathbb{C}\langle X\rangle * \mathbb{C}\langle Y\rangle = \mathbb{C}\langle X, Y\rangle$ (noncommuting polynomials in $X$ and $Y$). The conditionally free convolution

$$\mu = (\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$$

is the compactly supported measure on $\mathbb{R}$ that corresponds to the $\phi$-distribution of $X + Y$. We also construct $\nu = \nu_1 \boxplus \nu_2$ as the usual free convolution. This pair is denoted $(\mu, \nu) = (\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$.

**Definition 1.2.8.5.** The corresponding *conditionally free cumulant* functionals $R^{\phi,\psi}$ are recursively defined

$$\phi[X_1, ..., X_n] = \sum_{\pi \in \mathrm{NC}(n)} \left( \prod_{\substack{V_i \in \pi \\ V_i \text{ inner}}} R^{\psi}_{V_i}[X_{\vec{V_i}}] \right) \left( \prod_{\substack{V_j \in \pi \\ V_j \text{ outer}}} R^{\phi,\psi}_{V_j}[X_{\vec{V_j}}] \right), \tag{1.8}$$

where $R^{\psi}$ denotes the usual free cumulant functionals with respect to $\psi$.

Then, in a similar fashion to free convolutions, the authors prove (using an induction argument) the conditionally free convolution $(\mu, \nu) = (\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$ can be described entirely in terms of the free and conditionally free cumulants of the two distributions:

$$R^{\nu}_n = R^{\nu_1}_n + R^{\nu_2}_n$$

and

$$R^{\mu,\nu}_n = R^{\mu_1,\nu_1}_n + R^{\mu_2,\nu_2}_n.$$

### 1.2.9 Operator-Valued Free Probability

In 1998, Roland Speicher introduced operator-valued distributions, with motivation from random (block) matrices ([18]). This section is primarily a reference to make the comparisons between our operator-valued objects and Speicher's more reader-friendly (for example, 2.2.4.1 is an "operator-valued free cumulant" of sorts, but not quite Speicher's definition). Therefore, this section will only contain the basic concepts.

**Definition 1.2.9.1.** Let $\mathcal{A}$ be a C*-algebra, with $\mathcal{B} \subset \mathcal{A}$ a sub-C*-algebra. A *conditional expectation* $\mathbb{E} : \mathcal{A} \to \mathcal{B}$ is a map that satisfies

- $a \geq 0$ implies $\mathbb{E}[a] \geq 0$ (*positivity*), and
- $\mathbb{E}[b_1 a b_2] = b_1 \mathbb{E}[a] b_2$ (*$\mathcal{B}-$bimodule property*).

Then $(\mathcal{A}, \mathcal{B}, \mathbb{E})$ is a *$\mathcal{B}$-valued non-commutative probability space*.

**Example 1.2.9.2.** The definition above was intended to generalize the following example to non-commutative operator algebras. Let $(X, \Sigma, P)$ be a probability measure space, with a sub-$\sigma$-algebra $\Theta \subset \Sigma$. Then let $\mathcal{A}$ be the $\Sigma$-measurable functions and $\mathcal{B}$ be the $\Theta$-measurable functions. The Radon-Nikodym theorem gives a conditional expectation $\mathbb{E} : \mathcal{A} \to \mathcal{B}$.

Distributions are not defined on non-commuting polynomials of scalar coefficients, but $\mathcal{B}$-*valued* coefficients. To be more precise, this space is the span of $b_0xb_1x...xb_n$, for $n \geq 0$, $b_i \in \mathcal{B}$, and $x$ an indeterminate that does not commute with $\mathcal{B}$. This space is denoted $\mathcal{B}\langle x \rangle$.

**Definition 1.2.9.3.** For a variable $X \in (\mathcal{A}, \mathbb{E})$ ($\mathcal{B}$-valued), its *distribution* is the $\mathcal{B}$-valued map $\mu_X : \mathcal{B}\langle x \rangle \to \mathcal{B}$ defined by

$$\mu_X[b_0xb_1x...xb_n] = \mathbb{E}[b_0Xb_1X...Xb_n] = b_0\mathbb{E}[Xb_1X...X]b_n.$$

Joint distributions of $X_1, ..., X_n$ are defined similarly, where the $X_i$ are in place of each $X$, in index order.

**Definition 1.2.9.4.** Inductively define the *$\mathcal{B}$-valued free cumulant* maps $R[b_0X_1b_1, X_2b_2, ..., X_nb_n] = b_0R[X_1b_1, X_2b_2, ..., X_n]b_n$ by

$$\mathbb{E}[b_0X_1b_1X_2b_2...X_nb_n] = \sum_{\pi \in \mathrm{NC}(n)} R_\pi[b_0X_1b_1, X_2b_2, ..., X_nb_n],$$

where $R_\pi$ uses the nesting structure on NC$(n)$ (cumulants corresponding to inner blocks are nested inside the cumulants of the outer blocks), as demonstrated in the following example.

**Example 1.2.9.5.** For $\pi = \{(15)(234)(6)\}$, we have

$$R_\pi[b_0X_1b_1, X_2b_2, X_3b_3, X_4b_4, X_5b_5, X_6b_6] = b_0R[X_1b_1R[X_2b_2, X_3b_3, X_4]b_4X_5]b_5R[X_6]b_6,$$

where we use the bimodule property of $R$ it inherited from $\mathbb{E}$.

**Definition 1.2.9.6.** Subalgbras $\mathcal{A}_1, ..., \mathcal{A}_n \subset (\mathcal{A}, \mathbb{E})$ ($\mathcal{B}$-valued) are *freely independent (with amalgamation) over* $\mathcal{B}$ is whenever

- $u(1) \neq u(2), u(2) \neq u(3), ..., u(k-1) \neq u(k)$ and
- $a_i \in \mathcal{A}_{u(i)}$ with $\mathbb{E}[a_i] = 0$,

then $\mathbb{E}[a_1...a_k] = 0$. As usual, elements are called freely independent (with amalgamation) if the subalgebras generated by them are.

**Theorem 1.2.9.7.** *Elements are freely independent (with amalgamation) over* $\mathcal{B}$ *if and only if their* $\mathcal{B}$*-valued free cumulants vanish.*

### 1.2.10 Combinatorics of Lattice Paths

Our conventions for *lattice paths* are as follows. These are piece-wise linear paths whose vertices are in $(\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$. The second coordinate will be called the *height*. Each path will start at $(0, 0)$ and will never go below a height of 0. Each step that we will consider is one of the following:

- a *rising* step: one unit to the right, one unit up,

- a *flat* step: one unit to the right, no height change,

- a *falling* step: one unit to the right, one unit down.

When we refer to a step's height, we mean its starting height, not ending.

**Definition 1.2.10.1.** *Dyck paths* are lattice paths that start and end at height 0, and consist of only rising and falling steps as defined above.

Also denote by $\mathrm{NC}_2(2n)$ the set of non-crossing *pairings*, partitions with only blocks of size 2.

**Lemma 1.2.10.2.** *There is a bijection between the set of Dyck paths with* $2n$ *steps and* $NC_2(2n)$.

*Proof.* Rising steps correspond to opening elements of blocks, falling steps to closing elements. ☐

**Definition 1.2.10.3.** *Motzkin paths* are lattice paths starting and ending at height 0, consisting of rising, falling, and flat steps as defined above.

**Lemma 1.2.10.4.** *Assign to each flat step a label $s$ or $m$, except those at height 0, which may only have the $s$ label. There is a bijection between the set of such labeled Motzkin path with $n$ steps and NC$(n)$.*

*Proof.* The proof is the same, but with $m$-flat steps corresponding to middle elements of a block, and $s$-flat steps corresponding to singleton blocks. ☐

**Remark 1.2.10.5.** For a partition $\pi$, the number of *open blocks* at position $i$ is

$$|\{V \in \pi \mid \min(V) < i \leq \max(V)\}|.$$

Then through either of the above bijections, the height of a path at step $i$ corresponds to the number of open blocks at $i$.

### 1.2.11 Fock Spaces

Introduced by V.A. Fock in 1932 as the quantum states space of multiple identical particles, Fock spaces have since seen use in a plethora of fields, including quantum mechanics, representation theory, combinatorics, and operator algebras. Our particular interest in Fock spaces stems from the potential to represent many classes of distributions via certain bounded operators (non-commuting random variables) on a Fock space. Fock spaces are appealing for this due to the nice combinatorial interpretations of those operators' distributions.

**Remark 1.2.11.1.** Here, I must clarify that by distribution, I am referring to a state on $\mathbb{C}\langle X_1, ..., X_n\rangle$, and by *representation*, I mean in the sense of Definition 1.2.2.2, in which a polynomial passes to a state of another *-algebra by replacing the indeterminates with respective elements of a family of non-commuting random variables. Thus, this representation question asks whether or not a given state on $\mathbb{C}\langle X_1, ..., X_n\rangle$ can be reconstructed by pulling back the state of another *-probability space to $\mathbb{C}\langle X_1, ..., X_n\rangle$.

**Construction 1.2.11.2.** (**Full Fock Space**) Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The *(algebraic) Fock space* of $\mathcal{H}$ is

$$\mathcal{F}_{\text{alg}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^{\otimes 0} := \mathbb{C}\Omega$, the span of the *vacuum vector*.

Define the inner product $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ by linear extension of

$$\langle f_1 \otimes ... \otimes f_n, g_1 \otimes ... \otimes g_k\rangle_{\mathcal{F}} = \delta_{n=k} \prod_{i=1}^{n} \langle f_i, g_i\rangle.$$

Note that on each $\mathcal{H}^{\otimes n}$, this is just the canonical tensor inner product. We simply require that each of those subspaces for different $n$ be orthogonal.

Let $\mathcal{F}(\mathcal{H}) :=$ denote the completion of $\mathcal{F}_{\text{alg}}(\mathcal{H})$ with respect to this inner product.

On $\mathcal{F}$, for $f \in \mathcal{H}_{\mathbb{R}}$ and $g_1, ..., g_n \in \mathcal{H}$, define the following operators:

- The **free creation operator** is given by linear extension of

$$a^{+}(f)\Omega = f,$$

$$a^{+}(f)(g_1 \otimes ... \otimes g_n) = f \otimes g_1 \otimes ... \otimes g_n.$$

- The **free annihilation operator** is given by linear extension of

$$a^{-}(f)\Omega = 0,$$

$$a^{-}(f)(g_1 \otimes ... \otimes g_n) = \langle f, g_1\rangle g_2 \otimes ... \otimes g_n.$$

**Lemma 1.2.11.3.** *Let $X(f) = a^{+}(f) + a^{-}(f)$. This is bounded and self-adjoint under the inner* *product of $\mathcal{F}(\mathcal{H})$.*

*Proof.* A quick calculation shows that $a^{+}(f)$ is bounded with operator norm $\leq \|f\|$. A second calculation shows that its adjoint is $a^{-}(f)$. $\square$

**Definition 1.2.11.4.** On $\mathcal{B}(\mathcal{F}(\mathcal{H}))$, define the *vacuum state* $\Phi[A] = \langle A\Omega, \Omega\rangle$. Then (Alg$\{X(f) \mid f \in \mathcal{H}\}, \Phi$) is a *-probability space.

Now, we explore the distributions we can represent with this construction.

**Proposition 1.2.11.5.** *For $n$ even and $f_1, ..., f_n \in \mathcal{H}$, we have*

$$\langle X(f_1)...X(f_n)\Omega, \Omega\rangle_{\mathcal{F}} = \sum_{\pi \in NC_2(n)} \langle W_M(\pi)\Omega, \Omega\rangle = \sum_{\pi \in NC_2(n)} \prod_{V \in \pi} \langle f_{V(1)}, f_{V(2)}\rangle \tag{1.9}$$

*where $W_M(\pi) = \prod_{i=1}^{n} a_i$, and*

$$a_i = \begin{cases} a^{+}(f_i), & \textit{if } i \textit{ is a closing element,} \\ a^{-}(f_i), & \textit{if } i \textit{ is an opening element.} \end{cases}$$

*All mixed moments for $n$ odd are zero.*

By Lemma 1.2.10.2, we can restate this in terms of weighted Dyck paths:

**Corollary 1.2.11.6.** *For $n$ even and $f_1, ..., f_n \in \mathcal{H}$, we have*

$$\langle X(f_1)...X(f_n)\Omega, \Omega\rangle_{\mathcal{F}} = \sum_{path \in Dyck(n)} \langle W_M(path)\Omega, \Omega\rangle \tag{1.10}$$

*where $W_M(path) = \prod_{i=1}^{n} a_i$, and*

$$a_i = \begin{cases} a^{+}(f_i), & \textit{if there is a falling step at } i, \\ a^{-}(f_i), & \textit{if there is an rising step at } i. \end{cases}$$

*All mixed moments for $n$ odd are zero.*

**Corollary 1.2.11.7.** *For $f_1, ..., f_n \in \mathcal{H}$, all free cumulants are zero except*

$$R[X(f_i), X(f_j)] = \langle f_i, f_j \rangle.$$

Hence, $X(f)$ has a semi-circular distribution with variance $\|f\|^2$.

**Corollary 1.2.11.8.** *If $f_1, ..., f_n \in \mathcal{H}$ are pairwise orthogonal, then $X(f_1), ..., X(f_n)$ are freely independent. In this case, these variables form a* ***free semi-circular system****.*

Next, we can look at the distributions of the $X(f_i)$ from the perspective of *orthogonal polynomials* in $\mathbb{C}\langle X_1, ..., X_n \rangle$.

**Definition 1.2.11.9.** For $f_1, ..., f_n \in \mathcal{H}_{\mathbb{R}}$, their *Wick product* (or *Wick polynomial*) $W(f_1, f_2, ..., f_n) \in \mathcal{B}(\mathcal{F}(\mathcal{H}))$ is defined recursively via

$$\begin{aligned} W(\emptyset) &= 1 \\ W(f_1) &= X(f_1) \\ W(f_1, f_2, ..., f_n) &= X(f_1)W(f_2, ..., f_n) - \langle f_2, f_1 \rangle W(f_3, ..., f_n). \end{aligned}$$

**Lemma 1.2.11.10.** *$W(f_1, ...f_n)$ is the only operator in the *-algebra generated by $\{X(f) \,|\, f \in \mathcal{H}_{\mathbb{R}}\}$ with the property*

$$W(f_1, ...f_n)\Omega = f_1 \otimes ... \otimes f_n.$$

To prove this, we need the following:

**Lemma 1.2.11.11.** *$\Omega$ is separating for this *-algebra, i.e., the only operator $X$ in the algebra such that $X\Omega = 0$ is $X = 0$.*

*Proof.* Take $X := \sum_{i=1}^{N} X(f_1^{(i)})...X(f_{n(i)}^{(i)})$ in the *-algebra, and assume $X\Omega = 0$. Consider

$$X\Omega = \sum_{j=0}^{M} g_j,$$

for some (possibly non-simple) tensors $g_j \in (\mathcal{H}^\circ)^{\otimes j}$, and where $M := \max n_i$ (the length of the longest word in $X$.)

Since $X\Omega = 0$, we have $g_j = 0$ for all $j$ by orthogonality. In particular,

$$g_M = \sum_{k=1}^{\ell} f_1^{(k_\ell)} \otimes ... \otimes f_M^{(k_\ell)} = 0,$$

where $k_\ell$ are such that $n_{k_\ell} = M$. This gives

$$G_M := \sum_{k=1}^{\ell} X(f_1^{(k_\ell)})...X(f_M^{(k_\ell)}) = 0.$$

Repeat this process for the longest tensors in $X\Omega - G_M\Omega$ to see that $X \equiv 0$. □

*Proof.* (**of Lemma 1.2.11.10**)

The equation follows by induction via the recursion. Uniqueness follows from the previous lemma. □

Next, for $f \in \mathcal{H}_\mathbb{R}$ with $\|f\|^2 = t$, denote $W_n(f) := W(f, ..., f)$ ($n$ times). It satisfies the recursion

$$W_{n+1}(f) = XW_n(f) - tW_{n-1}(f),$$

where $X := X(f)$. These correspond to the polynomial recursion

$$V_{n+1}(x) = xV_n(x) - tV_{n-1}(x).$$

Compare to the *Chebyshev polynomials of the second kind*:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

**Proposition 1.2.11.12.** $\{V_n\}_{n\in\mathbb{N}}$ *are orthogonal polynomials with respect to the semi-circular dis-*

*tribution $\mu$ with mean 0 and variance $t$, that is,*

$$\phi[W_m(f)^* W_n(f)] = \int_{\mathbb{R}} V_n(x) V_m(x) d\mu(x) = 0 \textit{ for } n \neq m.$$

*Proof.*

$$\begin{aligned}\int_{\mathbb{R}} V_n(f) V_m(f) d\mu(x) &= \langle W_n(f) W_m(f)\Omega, \Omega\rangle = \langle W_m(f)\Omega, W_n(f)\Omega\rangle \\ &= \langle f^{\otimes n}, f^{\otimes m}\rangle = \begin{cases} 0, & \text{if } n \neq m \\ t^n & \text{if } n = m. \end{cases}\end{aligned}$$

□

As we mentioned in the introduction, the three motivating constructions generalized the full Fock space in two ways:

- They utilized a deformed inner product, via a bilinear map of some sort whose arguments are adjacent components of the tensor (this map also appears in their respective annihilation operators), and
- They included a third operator in $X(f)$, which we denote $a^0(f)$, since it maps $\mathcal{H}^{\otimes n}$ to $\mathcal{H}^{\otimes n}$ in each construction.

## 2.1 The Primary Construction

**Construction 2.1.0.1.** To begin, let $\mathcal{B}$ be a unital *-algebra, equipped with star-linear maps $\phi : \mathcal{B} \to \mathbb{C}$, $\gamma : \mathcal{B} \to \mathcal{B}$, and $\Lambda : \mathcal{B} \otimes_{\text{alg}} \mathcal{B} \to \mathcal{B}$ such that $\phi$ is positive and faithful, and $\gamma + \phi$ is completely positive. On the algebraic Fock space $\mathcal{F}_{alg}(\mathcal{B})$,

$$\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{B}^{\otimes n},$$

define the inner product by the linear extension of

$$\begin{aligned} &\langle u_1 \otimes \ldots \otimes u_n, v_1 \otimes \ldots \otimes v_k \rangle_{\gamma,\phi} \\ &\qquad = \delta_{n=k} \phi \left[ v_n^* (\gamma + \phi) [v_{n-1}^* (\gamma + \phi) [\ldots (\gamma + \phi) [v_1^* u_1] \ldots] u_{n-1}] u_n \right]. \end{aligned} \tag{2.1}$$

This inner product is positive semi-definite but, in many cases, will not be positive definite. In that case, we would take the quotient space $\mathcal{F}_{alg}(\mathcal{B})/\mathcal{N}$, where $\mathcal{N}$ is the subspace for which the semi-norm induced by this inner product is zero. Denote the completion of this space by $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$.

For reasons that will be explained momentarily, we will assume the following relations throughout:

$$\phi[v^* \Lambda(b \otimes u)] = \phi[\Lambda(b^* \otimes v)^* u], \text{ and } \gamma[v^* \Lambda(b \otimes u)] = \gamma[\Lambda(b^* \otimes v)^* u], \tag{2.2}$$

**Remark 2.1.0.2.** Instead of the map $\gamma$, it suffices to use a bilinear map $\langle \cdot, \cdot \rangle_\gamma : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ which is star-linear in each argument and such that $\langle \cdot, \cdot \rangle_\gamma + \langle \cdot, \cdot \rangle_\phi$ is positive semi-definite.

**Lemma 2.1.0.3.** *If for some $t < 1$, $\gamma + t\phi$ is still completely positive, then the inner product* (2.1) *is non-degenerate.*

*Proof.* For $n \geq 2$, suppose

$$
\begin{aligned}
0 &= \left\langle \sum_i u_i^{(1)} \otimes \ldots \otimes u_i^{(n)}, \sum_i u_i^{(1)} \otimes \ldots \otimes u_i^{(n)} \right\rangle \\
&= \sum_{ij} \phi \left[ u_i^{(n)*}(\gamma+\phi) \left[ ...u_i^{(2)*}(\gamma+\phi) \left[ u_i^{(1)*} u_j^{(1)} \right] u_j^{(2)} ... \right] u_j^{(n)} \right] \\
&= \sum_{ij} \phi \left[ u_i^{(n)*}(\gamma+t\phi) \left[ ...u_i^{(2)*}(\gamma+t\phi) \left[ u_i^{(1)*} u_j^{(1)} \right] u_j^{(2)} ... \right] u_j^{(n)} \right] \\
&+ \sum_{ij} (\text{scalar multiples of products of multiple lesser tensor-power } (\gamma, t\phi)\text{-inner products}) \\
&+ (1-t)^{n-1} \sum_{ij} \phi \left[ u_i^{(1)*} u_j^{(1)} \right] ...\phi \left[ u_i^{(n)*} u_j^{(n)} \right].
\end{aligned}
$$

The first term is always nonnegative by complete positivity and Lemma 3.5.3 in [18]. The middle terms are nonnegative for the same reasons plus the fact that the Schur product of positive matrices is positive (namely, the matrices whose $ij$th entry is the $\gamma, t\phi$-inner product of the $i$th tensor with some terms omitted with the $j$th tensor with terms omitted from the same positions). The last term is nonnegative by the Schur product property, and moreover, this sum is zero if and only if the matrix whose $ij$th term is $\phi \left[ u_i^{(1)*} u_j^{(1)} \right] ...\phi \left[ u_i^{(n)*} u_j^{(n)} \right]$ is zero entrywise, which only occurs if $u_i^{(1)} \otimes u_i^{(2)} \otimes ...u_i^{(n)} = 0$ since $\phi$ is faithful. $\square$

Next, for each $b \in \mathcal{B}$, consider densely defined operators

$$a^+(b)(u_1 \otimes \ldots \otimes u_n) = b \otimes u_1 \otimes \ldots \otimes u_n,$$

$$a^-(b)(u_1 \otimes \ldots \otimes u_n) = (\gamma+\phi)[bu_1]u_2 \otimes \ldots \otimes u_n,$$

$$a^-(b)(u_1) = \phi[bu_1]\Omega,$$

$$a^0(b)(u_1 \otimes \ldots \otimes u_n) = \Lambda(b \otimes u_1) \otimes u_2 \otimes \ldots \otimes u_n,$$

$$a^-(b)(\Omega) = a^0(b)(\Omega) = 0$$

and

$$X(b) = a^+(b) + a^-(b) + a^0(b).$$

These operators (and their products) are always defined on $\mathcal{F}_{alg}(\mathcal{B})$.

Denote

$$\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B},\phi) = \mathrm{Alg}(X(b) : b \in \mathcal{B}) = \mathrm{Alg}(X(b) : b \in \mathcal{B}^{sa})$$

and define on it the *vacuum state* $A \mapsto \langle A\Omega, \Omega\rangle$. Our main objective with the first phase, and an important component of the second, is to understand the distributions of the $X(b)$ operators with respect to the vacuum state.

**Lemma 2.1.0.4.** *If $X(b)$ is bounded with respect to the semi-norm over $\mathcal{F}_{alg}(\mathcal{B})$, then it is well-defined over $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$ and is also norm-bounded.*

*Proof.* Boundedness implies norm-zero vectors map to norm-zero vectors, so this is an immediate consequence of the linearity of the operators. □

In Section 2.4, we will examine conditions under which these operators are bounded, provided $\mathcal{B}$ is a C*-algebra. Let us set that question aside for now, and address conditions for $X(b)$ to be symmetric (self-adjoint if bounded).

**Proposition 2.1.0.5.** *First,*

$$\left\langle a^+(b)u_1 \otimes ... \otimes u_n, v_1 \otimes ... \otimes v_m \right\rangle_{\gamma,\phi} = \left\langle u_1 \otimes ... \otimes u_n, a^-(b^*)v_1 \otimes ... \otimes v_m \right\rangle_{\gamma,\phi}.$$

*By* (2.2)*, we also have*

$$\left\langle a^0(b)u_1 \otimes ... \otimes u_n, v_1 \otimes ... \otimes v_m \right\rangle_{\gamma,\phi} = \left\langle u_1 \otimes ... \otimes u_n, a^0(b^*)v_1 \otimes ... \otimes v_m \right\rangle_{\gamma,\phi}.$$

*In the case when $\Lambda(b \otimes u) = \Lambda(b)u$, for $\Lambda : \mathcal{B} \to \mathcal{B}$,* (2.2) *simplifies to* $(\Lambda(b))^* = \Lambda(b^*)$.

*Moreover, for self-adjoint $b$, $X(b)$ is symmetric.*

The proof is straightforward and so has been omitted.

Before I continue, I must highlight the three constructions which have inspired our own.

### 2.1.1 Example: Lenczewski, Sałapata (2007) [2]

Their construction is as follows: starting with $\mathcal{H} = L^2(\mathbb{R}_+)$ under Lebesgue measure, the free Fock space over $\mathcal{H}$ is

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n} \left( \simeq \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n) \right),$$

under the inner product defined in Section 1.10.

The authors define a weight function on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$w(s,t) = \begin{cases} p & \text{if } 0 < s < t, \\ q & \text{if } 0 < t < s, \\ 1 & \text{otherwise.} \end{cases}$$

**Example 2.1.1.1.** Using the notation of our construction, they use the maps

$$\gamma[f](s) = \int w(s,t) f(t) dt, \quad \phi[f] = \int f(t) d(t), \quad \Lambda \equiv 0.$$

to construct the operator $X(f) = a^+(f) + a^-(f)$. The corresponding $(\gamma, \phi)$ inner product is non-degenerate since $w + 1 \otimes 1$ is invertible.

The authors showed that, with respect to the vacuum state, such operators represent the *Kesten distributions* with parameters $p$ and $q$, whose densities are

$$f_{p,q}(x) = \frac{1}{\pi} \frac{\sqrt{2(p+q) - x^2}}{2 - (2 - p - q)x^2}, \quad -\sqrt{2(p+q)} \leq x \leq \sqrt{2(p+q)}.$$

For $p = 0$ and $q = 1$, you get the *arcsine distribution* as a familiar example.

Moreover, by taking the operator $\Lambda(b \otimes f)(x) = f(s) \int \big(w(s,t) + \delta_{s=t}\big) b(t)\, dt$ to construct

$a^0(f)$, the operators $X(f_t) = a^+(f_t) + a^0(f_t) + a^-(f_t)$ with $f_t = \chi_{[0,t)}$ represent a $(p,q)$-interpolation of the *free Poisson process*.

### 2.1.2 Example: Anshelevich (2007) [1]

**Example 2.1.2.1.** Their construction is as follows: begin with $\mathcal{H} = \mathbb{R}^d$ with the usual orthonormal basis $e_1, ..., e_d$. Then, again using our notation, they use the following maps (defined on the basis, and linearly extended)

$$\phi[e_j] = 1, \quad \gamma[e_j] = \sum_{i=1}^{d} C_{ij} e_i, \quad \Lambda(e_i \otimes e_j) = \sum_{k=1}^{d} B_{ij}^k e_k,$$

to construct the operator $X(f) = a^+(f) + a^0(f) + a^-(f)$. The natural embedding of $\mathbb{R}^d$ into $L^2(\mathbb{R})_+$ relates this example to the previous one.

Under the vacuum state, the author showed that these operators represent the *free Meixner distributions*, a large class which includes the (multi-variate versions of the) semicircular, free Poisson (Marchenko–Pastur), Bernoulli, and free binomial distributions.

Here is a rigorous definition of the free Meixner class. For a state $\phi$ on $\mathbb{R}\langle x_1, ..., x_d\rangle$, a *monic orthogonal polynomial system* is a subset $P$ of $\mathbb{R}\langle x_1, ..., x_d\rangle$ consisting of monic polynomials such that

- for any multi-index $\mathbf{u}$, there exists $P_{\mathbf{u}} \in P$ with leading term $x_{u(1)} ... x_{u(|\mathbf{u}|)}$,
- $\phi\big[(P_{\mathbf{u}})^* P_{\mathbf{v}}\big] = 0$ for $\mathbf{u} \neq \mathbf{v}$.

Anshelevich defined a free Meixner distribution to be a state $\phi$ on $\mathbb{R}\langle x_1, ..., x_d\rangle$ with a MOPS such that $\phi[x_i] = 0$ for all $i$, $R_\phi[x_i, x_j] = \delta_{i=j}$, and $\phi$ can be represented as the vacuum state on this Fock space construction for some symmetric operators $T_i := \Lambda(e_i \otimes \cdot)$ and a diagonal non-negative matrix $C$ with $(T_i \otimes I)C = C(T_i \otimes I)$.

### 2.1.3 Example: Bożejko, Lytvynov (2009) [3], [4]

The authors begin with $\mathcal{H} = L^2(X, \sigma)$ where $X$ is locally compact, second countable, Hausdorff, and contains no isolated points, and $\sigma$ is a non-atomic Radon measure on $(X, \text{Borel}(X))$.

**Example 2.1.3.1.** Assume, for our purposes, that $X$ is compact, and consider $\mathcal{B} = L^{\infty}(X, \sigma)$. Then take $\eta, \lambda \in \mathcal{B}$ such that $\eta$ is positive (i.e. non-negative a.e.) and $\lambda$ is self-adjoint (i.e. real-valued a.e.). Then with the maps

$$\gamma[f] = \eta f, \quad \Lambda(b \otimes f) = \lambda b f, \quad \phi[f] = \int_X f(t) d\sigma(t),$$

the authors define $X(f) = a^{+}(f) + a^{-}(f) + a^{0}(f)$ for $f : X \to \mathbb{R}$ continuous and compactly supported. They proved that these operators also represent the free Meixner distributions. Note that the $(\gamma, \phi)$ inner product here is non-degenerate if $\eta$ is invertible.

This setting generalizes somewhat to a general $*$-algebra $\mathcal{B}$ if we simply assume that $\eta$ and $\lambda$ are *central* elements of $\mathcal{B}$, that is, $\eta$ and $\lambda$ commute with all of $\mathcal{B}$.

### 2.1.4 Other Examples

Some slight generalizations of the $\Lambda$ in Example 2.1.1.1 are

$$\Lambda(b \otimes f)(t) = \int \lambda(s, x, y) b(x) f(y)\, dx\, dy \quad \text{or} \quad \Lambda(b \otimes f)(x) = f(s) \int \lambda(s, t) b(t)\, dt. \tag{2.3}$$

In the first case, conditions on $\Lambda$ in Proposition 2.1.0.5 correspond to $\lambda(s, x, y) = \overline{\lambda(y, x, s)}$ and

$$w(s, t)\lambda(t, x, y) = w(s, y)\lambda(t, x, y), \tag{2.4}$$

i.e. $\lambda(t, x, y) \neq 0$ implies $w(s, t) = w(s, y)$ for all $s$. In the second case, these conditions correspond to $\lambda$ real-valued.

The final example I will discuss is relatively simple, but important, since it includes the setting for our second phase.

**Example 2.1.4.1.** Choose $\mathcal{B}$, $\phi$, $\Lambda$ to be general, but let $\gamma = \psi$ be scalar-valued. Then the inner product simplifies to

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_k \rangle_{\gamma,\phi} = \delta_{n=k} \phi[g_n^* f_n] \prod_{i=1}^{n-1} (\psi + \phi)[g_i^* f_i].$$

Note that if $\mathcal{B} = L^\infty([0,1])$ and we are in the setting of Example 2.1.1.1, for absolutely continuous measures this corresponds to $w(u,v) = w(v)$. Then condition (2.4) implies that for non-zero $\lambda$, $w$ is constant, so $\gamma = \psi$ is a multiple of $\phi$.

## 2.2 Distributions Through Moments, Free Cumulants

### 2.2.1 Moments

In the moment and Boolean cumulant formulas below, given $\pi \in \mathrm{NC}(n)$ and $u_1, ..., u_n \in \mathcal{B}$, we will assign the following weight operator on $\mathcal{F}_{alg}(\mathcal{B})$:

$$W_M(\pi) = \prod_{i=1}^{n} a_i(u_i),$$

where

$$a_i = \begin{cases} a^+, \text{ if } i \text{ is a closing element,} \\ a^-, \text{ if } i \text{ is an opening element,} \\ a^0, \text{ if } i \text{ is a middle element.} \end{cases}$$

**Proposition 2.2.1.1.** *Given $u_1, ..., u_n \in \mathcal{B}$, we have the following mixed moment formula:*

$$\langle X(u_1)...X(u_n)\Omega, \Omega \rangle = \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega \rangle_{\gamma,\phi}, \tag{2.5}$$

*if $n \geq 2$, where $NC_{ns}(n)$ is the set of noncrossing partitions of $[n]$ with no singleton blocks. If $n = 1$, the moment is zero.*

*Proof.* By the definition of $X(u_i)$ and linearity, the product of the $X(u_i)$ is a sum in which each term is a product of creation, annihilation, and $a^0$ operators, with their respective functions always appearing in increasing index order; that is, each takes the form $a_1(u_1)...a_n(u_n)$, where $a_i$ is either $a^+$, $a^-$, or $a^0$.

Consider $a_1(u_1)...a_n(u_n)\Omega$. If there exists $i$ such that $a_i = a^-$ and

$$|\{j > i : a_j = a^-\}| \geq |\{j > i : a_j = a^+\}|,$$

this term will be zero. When taking the inner product of the term and $\Omega$, the product will be zero unless the term equals a multiple of $\Omega$, a case which only holds when

$$|\{j : a_j = a^-\}| = |\{j : a_j = a^+\}|.$$

This implies that the only terms which contribute to the sum are the inner products of operator products such that there are an equal number of creators and annihilators and when going from right to left, at no point will the number of annihilators exceed the number of creators. Finally, any product whose right-to-left evaluation will involve applying $a^0$ to a scalar of $\Omega$ will be zero by definition. Viewing this behavior through the well-known correspondence between non-crossing partitions and Motzkin paths (where the height at any point on the path indicates the number of opened, but not closed, blocks when moving right to left), through the weights given above, it's clear that the only possibly nonzero terms of the sum are those induced by noncrossing, nonsingleton partitions of $[n]$. $\square$

The original statement of the proposition was in terms of Motzkin paths, and though the proof is essentially the same, it seems more intuitive, in the author's opinion. We include it here for the interested reader:

**Corollary 2.2.1.2.** *Given $u_1, ..., u_n \in \mathcal{B}$, we have the following mixed moment formula:*

$$\langle X(u_1)...X(u_n)\Omega, \Omega\rangle = \sum_{path \in MP'(n)} \langle W_M(path)\Omega, \Omega\rangle_{\gamma,\phi}, \tag{2.6}$$

*if $n \geq 2$, where MP'$(n)$ is the set of Motzkin paths of length $n$ with no flat steps at height zero, and*

$$W_M(\mathit{path}) = \prod_{i=1}^{n} a_i(u_i),$$

*where*

$$a_i = \begin{cases} a^+, & \text{if there is a falling step at } i, \\ a^-, & \text{if there is a rising at } i, \\ a^0, & \text{if there is a flat step at } i. \end{cases}$$

*If $n = 1$, the moment is zero.*

Equivalence of this and Proposition 2.2.1.1 immediately follows from the bijection in 1.2.10.4.

The proof of the formula in terms of paths (from scratch) is the same as before, except we have a more visual description of what is happening:

- Paths are followed from right to left, though the terms *rising* and *falling* will still be applied as if they were drawn from left to right, for consistency.
- Whenever $a^+$ is applied, take a falling step (since we are going right to left, this is one step left, and one step up).
- Whenever $a^-$ is applied, take a rising step (this is one step left, and one step down).
- Whenever $a^0$ is applied, take a flat step.
- The key intuition is that the height at each step $i$ corresponds to the length of the tensor after all $X(u_j)$ for $j > i$ have been applied.

### 2.2.2 Boolean Cumulants

**Definition 2.2.2.1.** Recall that $\widetilde{\mathrm{NC}}(n) = \{\pi \in \mathrm{NC}(n) | 1 \overset{\pi}{\sim} n\}$. Similarly, let

$$\widetilde{\mathrm{NC}}_{ns}(n) = \{\pi \in \mathrm{NC}(n) | 1 \overset{\pi}{\sim} n \text{ and } \pi \text{ has no singleton blocks}\}.$$

**Lemma 2.2.2.2.** *For $n \geq 2$, given $u_1, ..., u_n \in \mathcal{B}$, we have the following mixed Boolean cumulant formula:*

$$B_n[X(u_1), ..., X(u_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi}, \tag{2.7}$$

*and for $n = 1$, the cumulant is zero.*

*Proof.* $n = 1$ is clear. Assume the result holds for all natural numbers less than some $n$, and take $u_1, ..., u_n \in \mathcal{B}$. By Proposition 2.2.1.1, we have

$$\sum_{\pi \in \mathrm{Int}(n)} B_\pi[X(u_1), ..., X(u_n)] = \langle X(u_1)...X(u_n)\Omega, \Omega\rangle = \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi}.$$

For convenience, denote by $\mathrm{NC}_{ns,mo}(n)$ the noncrossing, no-singleton partitions of $[n]$ that have more than one outer block. After isolating the $n$th cumulant (corresponding to the partition $\hat{1}_n$ consisting of a single block), we get

$$\begin{aligned} B_n[X(u_1), ..., X(u_n)] &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi} - \sum_{\pi \in \mathrm{Int}(n)\setminus\{\hat{1}_n\}} B_\pi[X(u_1), ..., X(u_n)] \\ &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi} - \sum_{\pi \in \mathrm{Int}(n)\setminus\{\hat{1}_n\}} \prod_{V \in \pi} \sum_{\sigma \in \widetilde{\mathrm{NC}}_{ns}(|V|)} \langle W_M(\sigma)\Omega, \Omega\rangle_{\gamma,\phi} \\ &= \sum_{\pi \in \mathrm{NC}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi} - \sum_{\sigma \in \mathrm{NC}_{ns,mo}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi} \\ &= \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_M(\pi)\Omega, \Omega\rangle_{\gamma,\phi}, \end{aligned}$$

where the second equality follows from the induction hypothesis. □

**Corollary 2.2.2.3.** *If $\gamma = -\phi$, $B_n[X(u_1), ..., X(u_n)] = \langle a^-(u_1)a^0(u_2)\ldots a^0(u_{n-1})a^+(u_n)\Omega, \Omega\rangle$.*

### 2.2.3 Free Cumulants

**Definition 2.2.3.1.** Define the operator $a^\sim_\gamma(b)$ ($b \in \mathcal{B}$) by linear extension of

$$a^\sim_\gamma(b)(u_1 \otimes \ldots \otimes u_n) = \gamma[bu_1]u_2 \otimes \ldots \otimes u_n \text{ for } n \geq 2,$$

$$a^{\sim}_{\gamma}(b)(u_1) = 0 \text{ for } n = 1, \text{ and}$$

$$a^{\sim}_{\gamma}(b)(\Omega) = 0.$$

The operator $a^{\sim}_{\phi}(b)$ is defined in a similar manner (for $n \geq 2$, apply $\phi$, and $a^{\sim}_{\phi}(b)(\Omega) = 0$), with one critical exception:

$$a^{\sim}_{\phi}(b)(u_1) = \phi[bu_1]\Omega \text{ for } n = 1.$$

Thus $a^{-}(b) = a^{\sim}_{\gamma}(b) + a^{\sim}_{\phi}(b)$.

In the free cumulant formula below, given $\pi \in \mathrm{NC}(n)$ and $u_1, ..., u_n \in \mathcal{B}$, we will assign the following weight operator on $\mathcal{F}_{alg}(\mathcal{B})$:

$$W_C(\pi) = \prod_{i=1}^{n} a_i(u_i),$$

where

$$a_i = \begin{cases} a^{+}, \text{ if } i \text{ is a closing element,} \\ a^{\sim}_{\gamma}, \text{ if } i \neq 1 \text{ and is an opening element,} \\ a^{\sim}_{\phi}, \text{ if } i = 1, \text{ or} \\ a^{0}, \text{ if } i \text{ is a middle element.} \end{cases}$$

**Proposition 2.2.3.2.** *For $n \geq 2$, given $u_1, ..., u_n \in \mathcal{B}$, we have the following mixed free cumulant formula:*

$$R_n[X(u_1), ..., X(u_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W_C(\pi)\Omega, \Omega\rangle_{\gamma,\phi}, \tag{2.8}$$

*and for $n = 1$, the cumulant is zero.*

*Proof.* By Theorem 1 in [14] and Lemma 2.2.2.2,

$$\begin{aligned} \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} R_{\pi}[X(u_1), ..., X(u_n)] &= B_n[X(u_1), ..., X(u_n)] \\ &= \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \sum_{\ell \in \text{Block labelings}(\gamma,\phi)} \langle W(\pi,\ell)\Omega, \Omega\rangle_{\gamma,\phi}, \end{aligned} \tag{2.9}$$

where $\ell$ is a labeling of the blocks of $\pi$ with either a $\gamma$ or $\phi$ such that the block containing 1 and $n$ is labeled $\phi$, and $W(\pi, \ell) = \prod_{i=1}^{n} a_i(u_i)$, where

$$a_i = \begin{cases} a^+, \text{ if } i \text{ is a closing element,} \\ a_\gamma^\sim, \text{ if } i \text{ is an opening element of a block labeled } \gamma, \\ a_\phi^\sim, \text{ if } i \text{ is an opening element of a block labeled } \phi, \\ a_0, \text{ if } i \text{ is a middle element.} \end{cases}$$

This splitting of terms follows from linearity and the fact that the opening of each block is weighted with $(\gamma + \phi)$ of a product involving the operator in that position.

Since the first moment (and thus free cumulant) is zero, for $n = 2$, we have

$$R_2[X(u_1), X(u_2)] = M_2[X(u_1)X(u_2)] = \langle a^-(u_1)a^+(u_2)\Omega, \Omega\rangle_{\gamma,\phi},$$

and for $n = 3$, we have

$$R_3[X(u_1), X(u_2), X(u_3)] = M_3[X(u_1)X(u_2)X(u_3)] = \langle a^-(u_1)a^0(u_2)a^+(u_3)\Omega, \Omega\rangle_{\gamma,\phi}.$$

For induction, assume for all natural numbers less than some $n$, the formula holds.

Let $\ll$ denote the partial order on NC$(n)$ described in Subsection 1.2.6. Then the inductive hypothesis implies the left-hand side of (2.9) equals

$$R_n[X(u_1), ..., X(u_n)] + \sum_{\pi \in \widetilde{\mathrm{NC}_{ns}}(n) \setminus \{1_n\}} \sum_{\sigma \ll \pi} \langle W(\sigma)\Omega, \Omega\rangle_{\gamma,\phi},$$

where each $\sigma$ is labeled such that for all $V \in \pi$, $\sigma|_V$ has its unique outer block labeled $\phi$, while the rest (that is, the inner blocks) are labeled $\gamma$.

By Belinschi and Nica's lemma (provided as Lemma 1.2.6.4 in this work), the above sum can be rewritten as

$$R_n[X(u_1),...,X(u_n)] + \sum_{\sigma \in \widetilde{\mathrm{NC}}_{ns}(n)\setminus\{1_n\}} \sum_{S \in S_\sigma} \langle W(\sigma, \ell_S)\Omega, \Omega\rangle_{\gamma,\phi},$$

where $\ell_S$ is the labeling constructed by giving blocks in $S$ the label $\phi$ and the rest $\gamma$. Subtracting the sum from both sides gives the result. □

**Corollary 2.2.3.3.** *If* $\gamma = 0$, $R_n[X(u_1),...,X(u_n)] = \langle a^-(u_1)a^0(u_2)\dots a^0(u_{n-1})a^+(u_n)\Omega, \Omega\rangle$.

Once again, the original statement of this formula was in terms of Motzkin paths, translated via the bijection in Lemma 1.2.10.4:

**Corollary 2.2.3.4.** *For* $n \geq 2$*, given* $u_1,...,u_n \in \mathcal{B}$*, we have the following mixed free cumulant formula:*

$$R_n[X(u_1),...,X(u_n)] = \sum_{\mathit{path} \in MP(n-2)} \langle a^-(u_1)W_C(\mathit{path})a^+(u_n)\Omega, \Omega\rangle_{\gamma,\phi}, \tag{2.10}$$

*where MP*$(k)$ *is the set of all (unlabeled) Motzkin paths of length* $k$*, and*

$$W_C(\mathit{path}) = \prod_{i=1}^{n} a_i(u_i),$$

*where*

$$a_i = \begin{cases} a^+, \text{ if there is a falling step at } i-1, \\ a^{\sim}_{\gamma}, \text{ if there is a rising step at } i-1, \\ a^0, \text{ if there is a flat step at } i-1. \end{cases}$$

*For* $n = 1$*, the cumulant is zero.*

**Example 2.2.3.5.** In the setting of Example 2.1.3.1,

$$R[(X(f_1),\dots,X(f_n)] = \sum_{\pi \in \widetilde{\mathrm{NC}}} \phi[\eta^{|\pi|-1}\lambda^{n-2|\pi|}f_1 f_2 \dots f_n].$$

In particular, in this case $R'[(X(f_2), \ldots, X(f_{n-1})]$ below may be identified with an element of $\mathcal{B}$.

Note also that if $fg = gf = f^*g = fg^* = 0$, then $X(f)$ and $X(g)$ are free, and if $f$ is self-adjoint with $\eta f = \lambda f = 0$, then $X(f)$ is semicircular.

See Section 2.5 for related results.

### 2.2.4 Cumulant-like Generating Function

We can define the $\mathcal{B}$-valued "kernel" of a free cumulant $R[u_1, ..., u_k]$ as follows:

**Definition 2.2.4.1.** For each $u_1, \ldots, u_k \in \mathcal{B}$, the (a priori unbounded) linear operator $R'[u_1, ..., u_k]$ on $\mathcal{B}$ is defined by

$$R'[u_1, ..., u_k] = \left( \sum_{\pi \in \mathrm{Int}(\{1,...,k\})} \prod_{V \in \pi} w(V) \right), \tag{2.11}$$

where the products are ordered by each block's appearance in the partition (from left to right), and the weights are given by $w(\{i\}) = a^0(u_i)$ and $w(\{i_1, ..., i_k\}) = \gamma[u_{i_1} R'[u_{i_2}, ..., u_{i_{k-1}}] u_{i_k}]$ for $k \geq 2$, with $R'[\emptyset] = 1$.

Note that for $\Lambda(u \otimes v) = \Lambda(u)v$, $R'[u_1, ..., u_k]$ is the operator of multiplication by an element of $\mathcal{B}$.

The definition may seem strange at first glance but will make sense after we prove the following property of this operator. Note that by Riesz's representation theorem, we could have simply defined $R'$ by this property instead.

**Lemma 2.2.4.2.**

$$R[X(u_1), \ldots, X(u_n)] = \langle R'[u_2, \ldots, u_{n-1}]u_n, u_1^* \rangle = \phi[u_1 R'[u_2, \ldots, u_{n-1}]u_n]. \tag{2.12}$$

*Proof.* To show that $\left\langle \left( \sum_{\pi \in \mathrm{Int}(\{2,...,n-1\})} \prod_{V \in \pi} w(V) \right) u_n, u_1^* \right\rangle = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} \langle W(\pi)\Omega, \Omega \rangle_{\gamma,\phi}$, we will prove a slightly stronger statement:

$$R'' := u_1 \left( \sum_{\pi \in \mathrm{Int}(\{2,...,n-1\})} \prod_{V \in \pi} w(V) \right) u_n = \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} W'(\pi)\Omega, \tag{2.13}$$

where the weights in the right-hand side are the same as in the cumulant formula, except all opening steps at 1 will be weighted by the identity map instead of $a^{\sim}_{\phi}$. In Example 2.1.1.1, the change is simply neglecting to integrate the last variable.

For $n = 2$, we have $u_1 u_2$ on both sides. For $n = 3$, we have $u_1 a^0(u_2) u_3$. For induction, assume the lemma holds for all natural numbers less than some $n$. Then

$$\sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} W'(\pi)\Omega = u_1 \left( \sum_{\pi \in \mathrm{Int}(\{2,...,n-1\})} \prod_{V \in \pi} w(V) \right) u_n,$$

where for $k \geq 2$, $w(\{i_1, ..., i_k\}) = \gamma \left[ u_{i_1} \left( \sum_{\pi \in \mathrm{Int}(\{2,...,k-1\})} \prod_{V \in \pi} w(V) \right) u_{i_k} \right]$.

At this point, we are done, since each $\pi \in \widetilde{\mathrm{NC}}_{ns}(n)$ can be uniquely constructed by taking some $\sigma \in \mathrm{Int}\{2, ..., n-1\}$, then constructing the unique outer block $\{1, \text{singletons}(\sigma), n\}$, then each nested block immediately below the outer block is recursively constructed in the same manner.

Finally, applying the vacuum state to both sides of (2.13) gives the result. $\square$

Next, we prove an equation which characterizes the generating function for this family of operators, which in turn will give us an equation for the free cumulant generating function.

**Lemma 2.2.4.3.** *Denote $R'_n[u] := R'[u, ..., u]$ (n arguments), where $R'_0[u] = 1$. Then*

$$R'_n[u] = \sum_{i=0}^{n-2} R'_i[u] \gamma[u R'_{n-i-2}[u] u] + R'_{n-1}[u] a^0(u). \tag{2.14}$$

*Proof.* The claim is analogous to the recursion for the number $I_n$ of interval partitions of length $n$,

$$I_n = \sum_{i=0}^{n-1} I_i,$$

where $n - i$ is the number of elements in the block containing $n$. The right-hand side in (2.14) is obtained in a similar manner, in which each term is obtained by collecting all terms in the sum (2.11) (over interval partitions) for $R'_n[u]$ for which the weight for the block containing $n$ is a factor. Thus, by summing over $i$ where $n - i$ is the number of elements in the block containing

$n$, the term corresponding to each $i$ is a product of $R'_i[u]$ and either $\gamma[uR'_{n-i-2}[u]u]$, the weight of the block containing $n$ for $i \leq n-2$, or $a^0(u)$, the weight of the singleton block containing $n$ for $i = n-1$. $\square$

**Theorem 2.2.4.4.** *Let $R'(u)$ be the generating function of $R'_n[u]$, $n \geq 0$. Then for any $v \in \mathcal{B}$,*

$$R'(u)v = v + R'(u)\gamma\left[uR'(u)u\right]v + R'(u)\Lambda(u \otimes v). \tag{2.15}$$

*Proof.* Applying Lemma 2.2.4.3,

$$\begin{aligned}
R'(u) &= 1 + \sum_{n=1}^{\infty} R'_n[u] \\
&= 1 + \sum_{n=1}^{\infty} \left( \sum_{i=0}^{n-2} R'_i[u]\gamma[uR'_{n-i-2}[u]u] + R'_{n-1}[u]a^0(u) \right) \\
&= 1 + \sum_{n=1}^{\infty} \left( \sum_{i=0}^{n-2} R'_i[u]\gamma[uR'_{n-i-2}[u]u] \right) + \sum_{n=1}^{\infty} R'_{n-1}[u]a^0(u) \\
&= 1 + \left( \sum_{n=0}^{\infty} R'_n[u] \right) \gamma \left[ u \sum_{n=0}^{\infty} R'_n[u]u \right] + R'(u)a^0(u) \\
&= 1 + R'(u)\gamma\left[uR'(u)u\right] + R'(u)a^0(u).
\end{aligned}$$

$\square$

**Corollary 2.2.4.5.** *More generally, for $u_1, ..., u_k \in \mathcal{B}$, let*

$$R'(u_1, ..., u_k) = \sum_{n=0}^{\infty} \sum_{|j|=n} R'_n[u_{j(1)}, ..., u_{j(n)}]$$

*be the generating function for the mixed $R'_n$. Then*

$$\begin{aligned}&R'(u_1, ..., u_k)v\\&= v + R'(u_1, ..., u_k)\gamma\left[\left(\sum_{i=1}^{k} u_i\right) R'(u_1, ..., u_k)\left(\sum_{j=1}^{k} u_j\right)\right] v + R'(u_1, ..., u_k)\sum_{i=1}^{k} \Lambda(u_i, v).\end{aligned} \tag{2.16}$$

*Proof.* Apply Theorem 2.2.4.4 to $R'\left(\sum_{i=1}^{k} u_i\right) = R'(u_1, ..., u_k)$. □

**Remark 2.2.4.6.** The generating function for $R''_n[u]$ is given by $R''(u) = uR'(u)u$, since $R''_n[u]$ is only defined for $n \geq 2$. Thus, $R''(u)$ satisfies

$$R''(u) = u^2 + uR'(u)\gamma\left[R''(u)\right]u + uR'(u)\Lambda(u, u). \tag{2.17}$$

Applying the vacuum state to both sides gives an equation for the cumulant generating function. Note that $R''(u)$ is a series of elements of $\mathcal{B}$.

**Example 2.2.4.7.** For $\Lambda(u \otimes v) = \Lambda(u)v$,

$$R'(u) = \left(1 - \Lambda(u) - \gamma[uR'(u)u]\right)^{-1}.$$

In particular, in the setting of Example 2.1.1.1, we may identify $R'(f)$ with a function satisfying

$$R'(f)(t) = \frac{1}{1 - \int \lambda(s,t)f(s)\,ds - f(t)\int w(s,t)R'(f)(s)f(s)\,ds}.$$

**Example 2.2.4.8.** In the setting of Example 2.1.3.1,

$$R'(f) = 1 + \lambda R'(f)f + \eta R'(f)fR'(f). \tag{2.18}$$

In the case where $\mathcal{B} = L^\infty([0,1])$, we may take $R'(f)$ to be the operator of pointwise multiplication by a function $R'(f)(x)$ which satisfies

$$R'(f)(x) = 1 + R'(f)(x)\lambda(x)f(x) + R'(f)(x)^2\eta(x)f(x),$$

and so is a solution of a quadratic equation for each $x$. We also have

$$R(f) = \int f(x)^2 R'(f)(x)\, dx.$$

**Remark 2.2.4.9.** Constructions and results in this section are reminiscent of operator-valued probability theory, such as those in [10]. In this remark we indicate how these constructions differ. The map

$$b_1 \otimes b_1 \otimes \ldots \otimes b_n \mapsto b_1 X b_2 X \ldots b_n X, \quad \Omega \mapsto 1_{\mathcal{B}}$$

is an isomorphism from the algebraic Fock space $\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$ onto a subspace of non-commutative polynomials $\mathcal{B}\langle X \rangle$. Using this identification, the relation between the operators in this article (for $\Lambda(f \otimes g) = \Lambda(f)g$) and in Proposition 3.1 from [10] is:

$$a^+(b) \leftrightarrow ba^*, \quad a^-(b) \leftrightarrow a\, b, \quad a^0(b) \leftrightarrow p$$

with $\alpha_1 = \phi$, $\alpha_n = \gamma + \phi$ for $n \geq 2$, and $\lambda_n = \Lambda(b)$ for $n \geq 1$. Thus the operator $X(1)$ here is the same as the $\mathcal{B}$-valued $X$, but the interaction with the algebra $\mathcal{B}$ is different in the two settings. Similarly, the identity

$$R'(b) = 1 + R'(b)\gamma\,[bR'(b)b] + R'(b)\Lambda(b).$$

satisfied by the (under appropriate assumptions) $\mathcal{B}$-valued generating function $R'(b)$ (from Theorem 2.2.4.4) is similar to, but different from the relation

$$b^{-1}R_\mu(b)b^{-1} = 1 + \gamma[R_\mu(b)b^{-1}]R_\mu(b)b^{-1} + \lambda R_\mu(b)b^{-1}.$$

from Proposition 3.22 in [10]. They do again coincide (up to a flip) for $b = 1$.

In light of the preceding remark, the following is related to Theorem 2 from [1].

**Proposition 2.2.4.10.** *Let $\mathcal{B}$ be a $C^*$-algebra and $u_1, \ldots, u_n \in \mathcal{B}$ self-adjoint. Assume that $\Lambda(u \otimes v) = \Lambda(u)v$. Let $X_1, \ldots, X_n$ be a $\mathcal{B}$-valued semicircular system with means $\Lambda(u_1), \ldots, \Lambda(u_n)$ and the covariance matrix $\eta : \mathcal{B} \to M_n(\mathcal{B})$ with $\eta_{ij}[b] = \gamma[u_i b u_j]$; the existence of such a system in some $\mathcal{B}$-valued non-commutative probability space $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ is guaranteed by [20]. Then*

$$R[X(u_{j(1)}), X(u_{j(2)}), \ldots, X(u_{j(k-1)}), X(u_{j(k)})] = \phi[u_{j(1)} \mathbb{E}[X_{j(2)} \ldots X_{j(k-1)}] u_{j(k)}].$$

## 2.3 Polynomial Generating Functions

### 2.3.1 Wick Polynomials

**Definition 2.3.1.1.** For $u_1 \otimes \ldots \otimes u_n \in \mathcal{B}^{\otimes n}$, define the operator $W(u_1 \otimes \ldots \otimes u_n)$ on the algebraic Fock space by the recursion

$$\begin{aligned} &W(b \otimes u_1 \otimes \ldots \otimes u_n) \\ &\quad = X(b) W(u_1 \otimes \ldots \otimes u_n) - W(a^0(b)(u_1 \otimes \ldots \otimes u_n)) - W(a^-(b)(u_1 \otimes \ldots \otimes u_n)) \\ &\quad = X(b) W(u_1 \otimes \ldots \otimes u_n) - W(\Lambda(b \otimes u_1) \otimes u_2 \otimes \ldots \otimes u_n) - W((\gamma + \phi)[b u_1] u_2 \otimes \ldots \otimes u_n) \end{aligned}$$

with the initial conditions

$$W(\emptyset) = I, \quad W(u_1) = X(u_1), \quad W(u_1 \otimes u_2) = X(u_1) W(u_2) - W(\Lambda(u_1 \otimes u_2)) - \phi[u_1 u_2].$$

It follows immediately that $W(u_1 \otimes ... \otimes u_n)$ is a polynomial (which we will also call a *Wick polynomial*) in the variables $\{X(u) : u \in \mathcal{B}\}$, although note that it typically is not in $\mathrm{Alg}\{X(u_1), ..., X(u_n)\}$. It is also clear from the recursion that

$$W(u_1 \otimes ... \otimes u_n)\Omega = u_1 \otimes ... \otimes u_n \text{ for all } u_1, ..., u_n \in \mathcal{B}. \tag{2.19}$$

In particular, we denote $W_n(u) = W(u^{\otimes n})$. From (2.19), we get orthogonality between polynomials of different degrees under the vacuum state. Since each $W(u_1 \otimes ... \otimes u_n) \in \Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, it follows that $\Omega$ is cyclic for this algebra.

For the moment, we establish the generating function, and the inverse used to recover it, as a formal object, as seen in the next proposition. In Section 2.4, we will give conditions under which these series are convergent and correspond to well defined operators.

**Proposition 2.3.1.2.** *Denote* $W(u) = 1 + \sum_{n=1}^{\infty} W_n(u)$ *and*

$$b(u) = 1 + \Lambda(u \otimes u)u^{-1} + (\gamma + \phi)[u^2].$$

*Note that for the case* $\Lambda(u \otimes v) = \Lambda(u)v$*,* $b(u) = 1 + \Lambda(u) + (\gamma + \phi)[u^2]$*. Then*

$$(b(u) - X(u))W(u) = b(u) - \phi[u^2].$$

*Proof.*

$$X(u)W_n(u) = W_{n+1}(u) + (\gamma + \phi)[u^2]W_{n-1}(u) + \Lambda(u \otimes u)u^{-1}W_n(u),$$

$$X(u)W_1(u) = W_2(u) + \phi[u^2] + \Lambda(u \otimes u)u^{-1}W_1(u),$$

$$X(u) = W_1(u).$$

So

$$X(u)W(u) = W(u) - 1 + \phi[u^2] + (\gamma + \phi)[u^2](W(u) - 1) + \Lambda(u \otimes u)u^{-1}(W(u) - 1).$$

□

### 2.3.2 Matricial Generating Functions

In this section, we will explore a means of recovering the multi-variable polynomials $\{W(u_n \otimes ... \otimes u_m)\}$ for $n \leq m$ and $\{u_n\}_{n=1}^{\infty}$ a sequence in $\mathcal{B}$. Rather than attempt to directly treat their generating function, we instead can compute the following matrix $W$ which will contain the polynomials arranged so that the diagonal $n$ places to the right of the main diagonal will contain all those with degree $n$. In the following proposition, $\mathcal{L}(\ell^2)$ denotes the algebra of linear (not necessarily bounded) operators, and $\{E_{ij}\}$ are the standard matrix units in it.

We can think of the maps $X, W$, etc., as non-linear maps from the vector space $\mathcal{B}^\infty = \bigoplus_{i=1}^\infty \mathcal{B}$ to the corresponding matrices of operators. Moveover, if we put on $\mathcal{B}^\infty$ the natural norm $\|(u_i)_{i=1}^\infty\| = \sup_i \|u_i\|$, on a sufficiently small ball the image of this map consists of bounded operators. This map itself is not bounded, but the map $\approxeq \mapsto W(\approxeq) - I$ is bounded on every small ball. The same is true of the free cumulant generating function.

**Proposition 2.3.2.1.** *Let $\{u_i : i \in \mathbb{N}\} \subset \mathcal{B}$. Define matrices with operator entries $\Phi \in \mathcal{L}(\ell^2)$, $\Gamma, A^0 \in \mathcal{B} \otimes \mathcal{L}(\ell^2)$, $X, W \in \mathcal{L}(\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \otimes \ell^2)$ as follows:*

$$\begin{aligned}
X &= \sum_{i=1}^\infty X(u_i) \otimes E_{i,i+1}, \\
W &= \sum_{i=1}^\infty \left( E_{i,i} + \sum_{j=1}^\infty W(u_i \otimes \ldots \otimes u_{i+j-1}) \otimes E_{i,i+j} \right), \\
\Phi &= \sum_{i=1}^\infty \phi[u_i u_{i+1}] \otimes E_{i,i+2}, \\
\Gamma &= \sum_{i=1}^\infty \gamma[u_i u_{i+1}] \otimes E_{i,i+2}, \\
A^0 &= \sum_{i=1}^\infty \Lambda(u_i \otimes u_{i+1}) u_{i+1}^{-1} \otimes E_{i,i+1}.
\end{aligned}$$

*Denote $B = I + A^0 + \Gamma + \Phi \in \mathcal{L}(\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \otimes \ell^2)$. Then*

$$(B - X)W = (B - \Phi).$$

*Proof.* For $k > i+2$,

$$X(u_i)W(u_{i+1} \otimes \ldots \otimes u_{k-1}) = W(u_i \otimes \ldots \otimes u_{k-1}) + \Lambda(u_i \otimes u_{i+1})u_{i+1}^{-1}W(u_{i+1} \otimes \ldots \otimes u_{k-1}) + (\gamma[u_i u_{i+1}] + \phi[u_i u_{i+1}])W(u_{i+2} \otimes \ldots \otimes u_{k-1}).$$

For $k = i+2$,

$$X(u_i)W(u_{i+1}) = W(u_i \otimes u_{i+1}) + \Lambda(u_i \otimes u_{i+1})u_{i+1}^{-1}W(u_{i+1}) + \phi[u_i u_{i+1}].$$

For $k = i+1$,

$$X(u_i)1 = W(u_i)$$

By comparing matrix entries, we can see that this implies

$$XW = (W - I) + A^0(W - I) + \Phi + (\Gamma + \Phi)(W - I). \tag{2.20}$$

□

Since $B - X$ is an upper-triangular matrix with only 1s along the main diagonal, its inverse exists for a finite family of $\{u_i\}$, and in a strictly formal sense for an infinite family. In the next section, we will discuss conditions under which $(B - X)^{-1}$ (and $W$) is a bounded operator on $\mathcal{F}_{\gamma,\phi}(\mathcal{B}) \otimes \ell^2$.

Since $W$ contains all the information about the multivariate $W$'s, it be considered as a kind of generating function for them. By constructing certain corresponding operators on a matricial Fock space construction, we can view $W$ as a genuine generating function of these operators.

**Definition 2.3.2.2.** Let $\mathcal{D}$ be a $*$-algebra; below we will take $\mathcal{D} = \mathcal{L}(\ell^2)$. $\mathcal{B} \otimes \mathcal{D}$ is naturally a

$\mathcal{D}$-bimodule. On the Fock space

$$\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n} \otimes \mathcal{D} \simeq (\mathcal{B} \otimes \mathcal{D})^{\otimes_{\mathcal{D}} n},$$

we consider operators indexed by elements of $\mathcal{B} \otimes \mathcal{D}$

$$a^{+}(b \otimes d)(u_1 \otimes \ldots \otimes u_n \otimes d') = b \otimes u_1 \otimes \ldots \otimes u_n \otimes dd',$$

$$a^{-}(b \otimes d)(u_1 \otimes \ldots \otimes u_n \otimes d') = \gamma[bu_1]u_2 \otimes \ldots \otimes u_n \otimes dd',$$

$$a^{-}(b \otimes d)(u_1 \otimes d') = \phi[bu_1]dd',$$

$$a^{0}(b \otimes d)(u_1 \otimes \ldots \otimes u_n \otimes d') = \Lambda(b, u_1) \otimes u_2 \otimes \ldots \otimes u_n \otimes dd',$$

$$a^{-}(b \otimes d)(d') = a^{0}(b \otimes d)(d') = 0.$$

Then any operator in the algebra generated by $\{a^{+}(b \otimes d), a^{-}(b \otimes d), a^{0}(b \otimes d) : b \in \mathcal{B}, d \in \mathcal{D}\}$ is of the form $A \otimes d$, where $A$ acts purely on $\bigoplus_{n=0}^{\infty} \mathcal{B}^{\otimes n}$. Moreover, the map

$$\Psi : (A \otimes d) \mapsto \langle (A \otimes d)(1_{\mathcal{B}} \otimes 1_{\mathcal{D}}), 1_{\mathcal{B}} \otimes 1_{\mathcal{D}} \rangle = \langle A1_{\mathcal{B}}, 1_{\mathcal{B}} \rangle \, d \tag{2.21}$$

is a $\mathcal{D}$-valued conditional expectation on the algebra generated by $\{X(b \otimes d) : b \in \mathcal{B}, d \in \mathcal{D}\}$.

**Remark 2.3.2.3.** Let $\mathcal{D} = \mathcal{L}(\ell^2)$ be the algebra of linear operators on $\ell^2$. Let $\{u_i : i \in \mathbb{N}\} \subset \mathcal{B}$, and define the matrix $U \in \mathcal{B} \otimes \mathcal{L}(\ell^2)$ by $U = \sum_{i \geq 1} u_i \otimes E_{i,i+1}$. Then the objects from Proposition 2.3.2.1 are in fact $\Phi = (\phi \otimes I)[U^2]$, $\Gamma = (\gamma \otimes I)[U^2]$, $A^0 = \tilde{\Lambda}[U \otimes U]$, where

$$\tilde{\Lambda}[(u \otimes S) \otimes (v \otimes T)] = \Lambda(u \otimes v) \otimes (ST),$$

and $X = a^+(U) + a^-(U) + a^0(U)$. If, in addition, $\mathcal{B}$ is a C*-algebra and

$$\sup_i \|u_i\| < \infty, \tag{2.22}$$

then $U \in \mathcal{B} \otimes \mathcal{B}(\ell^2)$, where $\mathcal{B}(\ell^2)$ denotes the algebra of bounded operators.

Before interpreting $W(U)$ as a generating function, we will first establish the analogous results for moments and cumulants. The formulas below are similar to, but once again different from, those in Section 6.3 in [9]. The first of these is straightforward.

**Lemma 2.3.2.4.** *For $\Psi$ defined in* (2.21) *for the case of $\mathcal{D} = \mathcal{L}(\ell^2)$,*

$$\Psi[X^n] = \sum_{i=1}^{\infty} \langle X(u_i)...X(u_{i+n-1})\Omega, \Omega\rangle \ E_{i,i+n}. \tag{2.23}$$

Given a non-commutative operator-valued probability space

$$(\mathrm{Alg}(X(b \otimes d) : b \in \mathcal{B}, d \in \mathcal{D} = \mathcal{L}(\ell^2)), \mathcal{D}, \Psi),$$

we may define $\mathcal{D}$-valued free cumulants $\mathcal{R}[d_0 X, d_1 X, \ldots, d_{n-1} X d_n]$ as in Chapter 4 of [18]. However, we will only be interested in these for $d_0 = d_1 = \ldots = d_n = 1$. In this case, we have the relation

**Corollary 2.3.2.5.**

$$\mathcal{R}_n[X, ..., X] = \sum_{i=1}^{\infty} R[X(u_i), ..., X(u_{i+n-1})] E_{i,i+n}. \tag{2.24}$$

*Proof.* For $n = 1$,

$$\mathcal{R}_1[X]_{i,j} = \Phi[X]_{i,j} = \phi[X(u_i)], \text{ for } j - i = 1.$$

For $n = 2$,

$$\Phi[X^2] = \mathcal{R}_2[X, X] + \mathcal{R}_1[X]^2,$$

so

$$\mathcal{R}_2[X,X] = \Phi[X^2] - \mathcal{R}_1[X]^2 = \Phi[X^2] - \Phi[X]^2,$$

then

$$\begin{aligned}\mathcal{R}_2[X,X]_{i,j} &= \phi[X(u_i)X(u_{i+1})] - \phi[X(u_i)]\phi[X(u_{i+1})] \\ &= R_2[X(u_i), X(u_{i+1})], \text{ for } j-i=2.\end{aligned}$$

Next, assume this is true for up to $n-1$. Then

$$\mathcal{R}_n[X,...,X] = \phi[X^n] - \sum_{\pi\in\mathrm{NC}(n)\setminus\widehat{1}_n} \mathcal{R}_\pi[X,...,X],$$

where $\mathcal{R}_\pi[X,...,X] = \prod_{V\in\pi}\mathcal{R}_{|V|}[X,...,X]$. Applying the inductive hypothesis, the matrices in the right-hand side are nonzero only for the $i,j$th entries such that $j-i=n$, and moreover, after summing the right-hand side, the $i,j$th entry (for such $i,j$) is the cumulant $R[X(u_i),...,X(u_{i+j-1})]$. $\square$

**Definition 2.3.2.6.** Define the operators $W_n(U)$ on the Fock space $\mathcal{F}_{\gamma,\phi}(\mathcal{B})\otimes\mathcal{B}(\ell^2)$ by the recursion

$$\begin{aligned}X(U)W_n(U) = W_{n+1}(U) + \delta_{n\geq1}\tilde{\Lambda}(U\otimes U)U^{-1}W_n(U) \\ + \Big(\delta_{n\geq2}(\gamma\otimes I)[U\otimes U] + (\phi\otimes I)[U\otimes U]\Big)W_{n-1}(U).\end{aligned}$$

Clearly $W_n(U)\Omega = U^{\otimes n}$ and its generating function $\sum_n W_n(U)$ satisfies equation (2.20), hence it equals $W$ entry-wise.

With this definition, we have a similar construction for the promised generating function of the polynomials.

**Proposition 2.3.2.7.** *In the setting of Proposition 2.3.2.1, denote by $W_n$ the matrix with a single non-zero diagonal, with entries $(W_n)_{i,i+n} = W(u_i\otimes\ldots\otimes u_{i+n-1})$. Then as in the proof of that* *proposition, for $n \neq 2$*

$$XW_n = W_{n+1} + A^0 W_n + (\Gamma + \Phi) W_{n-1},$$

*and so $W_n(U)$ can be identified with this matrix $W_n$. Note that if we only have finitely many non-zero entries $u_i$, boundedness condition* (2.22) *holds automatically.*

## 2.4 Norm Estimates and Convergence of Generating Functions

The next two results can be used to estimate the norm of $X(f)$. The following lemma is closely related to Lemma 4 in [21] and Lemma 1 in [22], although it is not stated in quite this form in either of those sources.

**Lemma 2.4.0.1.** *Let $\mathcal{H}$ be a Hilbert space, and $K$ a positive operator on it. Denote $\langle \xi, \eta \rangle_K = \langle \xi, K\eta \rangle$ the corresponding deformed inner product. Then for an operator $X$ on $\mathcal{H}$, denoting by $X^*$ its adjoint with respect to the deformed inner product, $\|X\|_K \leq \sqrt{\|X\| \|X^*\|}$.*

*Proof.* Denote by $X'$ the adjoint of $X$ under the usual inner product. Then

$$\langle KX^*\xi, \eta \rangle = \langle X^*\xi, K\eta \rangle = \langle \xi, KX\eta \rangle = \langle X'K\xi, \eta \rangle, \tag{2.25}$$

which implies $KX^*X = X'KX \geq 0$. Then

$$KX^*X(KX^*X)^* = KX^*X^2X^*K \leq \left\| X^*X^2X^* \right\| K^2. \tag{2.26}$$

Taking the square root of both sides, we have

$$KX^*X \leq \sqrt{\|X^*X^2X^*\|} K \leq \|X\| \|X^*\| K. \tag{2.27}$$

So

$$\begin{aligned}\langle X\xi, X\xi\rangle_K &= \langle \xi, X^*X\xi\rangle_K = \langle \xi, KX^*X\xi\rangle \\ &\leq \|X\|\,\|X^*\|\,\langle \xi, K\xi\rangle \\ &= \|X\|\,\|X^*\|\,\langle \xi, \xi\rangle_K.\end{aligned}$$

□

**Remark 2.4.0.2.** In Example 2.1.1.1,

$$\langle F, G\rangle_{\gamma,\phi} = \langle F, KG\rangle$$

where $K$ is the multiplication operator by

$$[(1+w)\otimes 1^{\otimes(n-2)}]\cdot[1\otimes(1+w)\otimes 1^{\otimes(n-3)}]\cdot\ldots\cdot[1^{\otimes(n-2)}\otimes(1+w)].$$

In the commutative case we may identify

$$K(s_1, s_2, \ldots s_n) = (1+w(s_1,s_2))\ldots(1+w(s_{n-1},s_n)).$$

In Example 2.1.4.1, $\left\langle \vec{\xi}, \vec{\eta}\right\rangle_{\gamma,\phi}$ is the standard inner product on $L^2(\mathcal{B},\phi)\otimes L^2(\mathcal{B},\psi+\phi)^{\otimes(n-1)}$.

For the remainder of this section, we assume that $\mathcal{B}$ is a $C^*$-algebra, and the maps $\phi : \mathcal{B}\to\mathbb{C}$, $\gamma : \mathcal{B}\to\mathcal{B}$, and $\Lambda : \mathcal{B}\times L^2(\mathcal{B},\phi)\to L^2(\mathcal{B},\phi)$ are bounded.

**Proposition 2.4.0.3.**

*a.* $\|a^+(b)\|_{\gamma,\phi} = \|a^-(b)\|_{\gamma,\phi} \leq \sqrt{\max\{\|\phi[b^*b]\|, \|(\gamma+\phi)[b^*b]\|\}} \leq \sqrt{\max\{\|\phi\|, \|\gamma+\phi\|\}}\ \|b\|.$

*b. In the special case* $\Lambda(b\otimes u_1) = \Lambda(b)u_1$ *with* $\Lambda : \mathcal{B}\to\mathcal{B}$*,*

$$\left\|a^0(b)\right\|_{\gamma,\phi} \leq \|\Lambda\|\ \|b\|.$$

*c. In the setting of Example 2.1.1.1, for* $w \in \mathcal{B} \otimes_{\min} \mathcal{B}$*,*

$$\left\| a^-(b) \right\|_{\gamma,\phi} \leq \|w\| \ \|b\| \,.$$

*For* $\|\Lambda(b \otimes b_1)\|_\phi \leq \|\Lambda\| \ \|b\| \ \|b_1\|_\phi$*,*

$$\left\| a^0(b) \right\|_{\gamma,\phi} \leq \|\Lambda\| \ \|b\| \,.$$

*For the commutative particular case, these correspond to* $\|w\|_\infty < \infty$ *and*

$$\iint \left( \int_0^1 |\lambda(s,x,y)| \ dx \right)^2 ds \, dy < \infty.$$

*d. In the setting of Example 2.1.4.1,*

$$\left\| a^0(b) \right\|_{\gamma,\phi} \leq \max \left( \|\Lambda(\cdot,\cdot)\|_{\mathcal{B}\otimes L^2(\mathcal{B},\phi)} , \|\Lambda(\cdot,\cdot)\|_{\mathcal{B}\otimes L^2(\mathcal{B},\psi+\phi)} \right) \|b\| \,.$$

*It follows that in these cases, the operators in Construction 2.1.0.1 are well-defined and bounded on* $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$*.*

*Proof.* Since tensors of different length in the Fock space are orthogonal, it suffices to estimate $\|a^+(b)\|_{\mathcal{F}}$ separately on tensors of fixed length. For $n = 0$,

$$\frac{\langle X(b)\Omega, X(b)\Omega \rangle_{\gamma,\phi}}{\langle \Omega, \Omega \rangle_{\gamma,\phi}} = \frac{\langle b, b \rangle_{\gamma,\phi}}{1} = \phi[b^2] = \|\phi\| \|b\|^2_{\mathcal{B}}.$$

For part (a) and $a^+$, let $u \in \mathcal{B}$. Then

$$\|a^+(b)u\|^2_{\gamma,\phi} = \phi[u^*(\gamma+\phi)[b^*b]u] \leq \|\gamma+\phi\|_{\mathcal{B}} \|b\|^2_{\mathcal{B}} \|u\|^2_{\gamma,\phi}.$$

For tensors of length $n \geq 2$, it suffices to consider a sum $\sum_{i=1}^k u_i^{(1)} \otimes ... \otimes u_i^{(n)}$ of simple tensors in $\mathcal{B}^{\otimes n}$, since $\mathcal{B}^{\otimes n}$ is the $\|\cdot\|_{\gamma,\phi}$-completion of the subspace of finite sums of simple tensors of length $n$.

$$
\begin{aligned}
&\left\| a^+(b) \sum_i u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 \\
&= \left\| \sum_i b \otimes u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 \\
&= \phi \left[ \sum_{ij} u_i^{(n)*} (\gamma + \phi) \left[ ...(\gamma + \phi) \left[ u_i^{(1)*} (\gamma + \phi)[b^* b] u_j^{(1)} \right] ... \right] u_j^{(n)} \right] \\
&\leq \|\gamma + \phi\|_{\mathcal{B}} \|b\|_{\mathcal{B}}^2 \phi \left[ \sum_{ij} u_i^{(n)*} (\gamma + \phi) \left[ ...(\gamma + \phi) \left[ u_i^{(1)*} u_j^{(1)} \right] ... \right] u_j^{(n)} \right] \\
&= \|\gamma + \phi\|_{\mathcal{B}} \|b\|_{\mathcal{B}}^2 \left\| \sum_i u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 ,
\end{aligned}
$$

where the inequality is due to positivity of the map

$$
x \mapsto \sum_{ij} z_i^* (\gamma + \phi) \left[ y_i^* x y_j \right] z_j
$$

for any $\{y_i\}, \{z_j\} \subset \mathcal{B}$, which is a result of complete positivity of $\gamma + \phi$, the positivity of the matrix $[y_i^* (\gamma + \phi)[b^* b] y_j]_{ij}$, and Lemma 3.5.3 in [18].

Hence, $\|a^+(b)\|_{\mathcal{F}} \leq \sqrt{\max\{\|\gamma + \phi\|_{\mathcal{B}}, \|\phi\|\}} \|b\|_{\mathcal{B}}$.

Since $a^-(b)$ is the adjoint of $a^+(b)$, their operator norms are equal, so the above inequality also applies to $a^\sim$.

For part (b) and $\Lambda(b)$ applied to tensors of length 1:

$$
\|\Lambda(b) u\|_{\gamma,\phi}^2 = \phi[u^* \Lambda(b)^* \Lambda(b) u] \leq \|\Lambda(b)\|_{\mathcal{B}}^2 \, \phi[u^* u] \leq \|\Lambda\|_{\mathcal{B} \to \mathcal{B}}^2 \, \|b\|_{\mathcal{B}}^2 \, \phi[u^* u].
$$

For $n \geq 2$:

$$\begin{aligned}
&\left\| \Lambda(b) \sum_i u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 \\
&= \left\| \sum_i \Lambda(b) u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 \\
&= \phi \left[ \sum_{ij} u_i^{(n)*} (\gamma + \phi) \left[ ...(\gamma + \phi) \left[ u_i^{(1)*} \Lambda(b)^* \Lambda(b) u_j^{(1)} \right] ... \right] u_j^{(n)} \right] \\
&\leq \|\Lambda\|_{\mathcal{B}\to\mathcal{B}}^2 \|b\|_{\mathcal{B}}^2 \phi \left[ \sum_{ij} u_i^{(n)*} (\gamma + \phi) \left[ ...(\gamma + \phi) \left[ u_i^{(1)*} u_j^{(1)} \right] ... \right] u_j^{(n)} \right] \\
&= \|\Lambda\|_{\mathcal{B}\to\mathcal{B}}^2 \|b\|_{\mathcal{B}}^2 \left\| \sum_i u_i^{(1)} \otimes u_i^{(2)} \otimes ... \otimes u_i^{(n)} \right\|_{\gamma,\phi}^2 ,
\end{aligned}$$

where the inequality follows in the same way as before.

For part (c), we apply Lemma 2.4.0.1. Part (d) follows from standard tensor product properties.

$\square$

**Proposition 2.4.0.4.** *Suppose* $\Lambda(f \otimes g) = \Lambda(f)g$. *Let* $u_1, \ldots, u_n \in \mathcal{B}$. *Define recursively a sequence* $(\alpha_j)$ *by* $\alpha_j = 2\alpha_{j-1} + \alpha_{j-2}$, $\alpha_0 = 1$, $\alpha_1 = 1$. *Then, denoting* $c_{\gamma,\phi} = \sqrt{\max\{\|\phi\|, \|\gamma + \phi\|\}}$,

$$\|W(u_1)\| \leq 2c_{\gamma,\phi} + \|\Lambda\| ,$$

*and for* $n \geq 2$ *and* $K = 2 + \frac{\|\Lambda\|}{c_{\gamma,\phi}} + \frac{1}{2c_{\gamma,\phi}^2}$,

$$\|W(u_1, \ldots, u_n)\| \leq \alpha_n c_{\gamma,\phi}^n K \|u_1\| \ldots \|u_n\| .$$

*Proof.* $W(u_1) = X(u_1)$,

$$W(u_1, u_2) = X(u_1)W(u_2) - \Lambda(u_1)W(u_2) - \phi[u_1 u_2] = (a^+(u_1) + a^-(u_1))W(u_2) - \phi[u_1 u_2],$$

and for $n \geq 3$,

$$W(u_1, u_2, \ldots, u_n) = (a^+(u_1) + a^-(u_1))W(u_2, \ldots, u_n) - (\gamma + \phi)[u_1 u_2]W(u_3, \ldots, u_n).$$

Therefore by Proposition 2.4.0.3, $\|W(u)\| = \|X(u)\| \leq 2c_{\gamma,\phi} + \|\Lambda\|$,

$$\|W(u_1, u_2)\| \leq 2\left\|a^+(u_1)\right\| \|W(u_2)\| + |\phi[u_1, u_2]| \leq 2c_{\gamma,\phi} \ \|u_1\| \|W(u_2)\| + \|u_1\| \ \|u_2\|,$$

and for $n \geq 3$,

$$\begin{aligned}\|W(u_1, u_2, \ldots, u_n)\| \\ &\leq 2c_{\gamma,\phi} \ \|u_1\| \|W(u_2, \ldots, u_n)\| + c_{\gamma,\phi}^2 \ \|u_1\| \ \|u_2\| \ \|W(u_3, \ldots, u_n)\|.\end{aligned}$$

The result follows by induction. □

Let's turn our attention back to the infinite matrix $W$ whose structure we established in the previous section, particularly the decomposition into a sum of matrices with a single non-zero diagonal. The following more general proposition will be combined with this fact to obtain conditions under which $W$ is a bounded operator. This result must be standard, although we do not have a reference for it.

**Lemma 2.4.0.5.** *Let $T \in \mathcal{B}(\mathcal{H} \otimes \ell^2)$, which we can write as an infinite matrix with entries $T_{ij}$. Then*

$$\|T\| \leq \sum_{i \in \mathbb{Z}} \sup_{k \geq \max(1, 1-i)} \|T_{k,k+i}\|.$$

*Proof.* Let $T_i$ be the infinite matrix with entries $(T_i)_{jk} = T_{jk}$ if $k - j = i$ and 0 otherwise (in other words, $T_i$ is comprised of the diagonal of $T$ which is $i$ positions above the main diagonal (below if negative)). Let $\sum_k u_k \otimes e_k \in \mathcal{H} \otimes \ell^2$. Then

$$\begin{aligned}
\left\| T_i \left( \sum_j u_j \otimes e_j \right) \right\|^2_{\mathcal{H}\otimes\ell^2} &= \left\| \sum_{j \geq \max(1,1-i)} T_{j,j+i} u_{j+i} \otimes e_j \right\|^2_{\mathcal{H}\otimes\ell^2} \\
&= \sum_{j \geq \max(1,1-i)} \| T_{j,j+i} u_{j+i} \otimes e_j \|^2_{\mathcal{H}\otimes\ell^2} \\
&\leq \sum_{j \geq \max(1,1-i)} \| T_{j,j+i} \|^2 \, \| u_{j+i} \otimes e_j \|^2_{\mathcal{H}\otimes\ell^2} \\
&\leq \left( \sup_{j \geq \max(1,1-i)} \| T_{j,j+i} \|^2 \right) \sum_{j \geq \max(1,1-i)} \| u_{j+i} \otimes e_j \|^2_{\mathcal{H}\otimes\ell^2} \\
&= \left( \sup_{j \geq \max(1,1-i)} \| T_{j,j+i} \|^2 \right) \left\| \sum_j u_j \otimes e_j \right\|^2_{\mathcal{H}\otimes\ell^2}
\end{aligned}$$

The result then follows by the triangle inequality. □

**Corollary 2.4.0.6.** *For* $\sup_i \|u_i\| \leq \frac{1}{(1+\sqrt{2})c_{\gamma,\phi}}$ *and* $\Lambda(u \otimes v) = \Lambda(u)v$*,*

$$\|W\| \leq \frac{K}{2\sqrt{2}} \frac{1}{1 - (1+\sqrt{2})c_{\gamma,\phi} \sup_i \|u_i\|},$$

*and*

$$W = (B - X)^{-1}(B - \Phi).$$

*Proof.* Solving the recursion,

$$\alpha_j = \frac{1}{2\sqrt{2}} \left( (1+\sqrt{2})^j - (1-\sqrt{2})^j \right).$$

The estimate on the norm of $W$ then follows from the preceding proposition, Lemma 2.4.0.5, and Proposition 2.3.2.1.

Next, let $Y = B - X - 1 = A^0 + \Gamma + \Phi - X$. We wish to show $(B - X)$ is invertible by checking that $\|Y\| < 1$. Note that $Y$ has non-zero entries on only two diagonals:

$$Y_{i,i+1} = a^{+}(u_i) + a^{-}(u_i)$$
$$Y_{i,i+2} = (\gamma + \phi)[u_i u_{i+1}].$$

So apply Lemma 2.4.0.5 and Proposition 2.4.0.3 to get the estimate

$$\begin{aligned}\|Y\| &\leq (\left\|a^{+}\right\| + \left\|a^{-}\right\|)\left(\sup_i \|u_i\|\right) + c_{\gamma,\phi}^2 \left(\sup_i \|u_i\|\right)^2 \\ &\leq 2c_{\gamma,\phi}\left(\sup_i \|u_i\|\right) + c_{\gamma,\phi}^2 \left(\sup_i \|u_i\|\right)^2,\end{aligned}$$

which is less than 1 if and only if $(\sup_i \|u_i\|) < \frac{\sqrt{2\sqrt{2}+3}}{c_{\gamma,\phi}}$ (satisfied by the assumption). □

Finally, we return to the question of convergence for the cumulant generating function.

**Proposition 2.4.0.7.** *Let* $K = \max\{\sqrt{\|\gamma\|}, \|\Lambda\|\}$. *Then* $\|R'_n[u]\| \leq C_n K^n \|u\|^n$, *where* $C_n$ *is the Catalan number. If* $\|u\| \leq \frac{1}{4K}$, *then the generating function* $R'(u)$ *converges, and thus the cumulant generating function does as well.*

*Proof.* By Lemma 2.2.4.3, $R'_0[u] = 1$, $\|R'_1[u]\| \leq \|\Lambda\| \|u\| \leq K \|u\|$, and

$$\|R'_2[u]\| \leq \|\Lambda\|^2 + \|\gamma\| \leq 2K^2 \|u\|^2.$$

Recursively,

$$\begin{aligned}\|R'_n[u]\| &\leq \sum_{i=0}^{n-2} \|R'_i[u]\| \left\|R'_{n-i-2}[u]\right\| \|\gamma\| \|u\|^2 + \left\|R'_{n-1}[u]\right\| \|\Lambda\| \|u\| \\ &\leq \sum_{i=0}^{n-2} C_i C_{n-i-2} K^{n-2} \|\gamma\| \|u\|^n + C_{n-1} K^{n-1} \|\Lambda\| \|u\|^n \\ &\leq C_n K^n \|u\|^n\end{aligned}$$

since $C_{n-1} = \sum_{i=0}^{n-2} C_i C_{n-i-2}$ and $C_n \geq 2C_{n-1}$. So the generating function has norm bounded by

$$\sum_{n=0}^{\infty} \|R'_n[u]\| \leq \sum_{n=0}^{\infty} C_n K^n \|u\|^n .$$

A well-known approximation of the Catalan numbers via Stirling's formula is $C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$. So this series' convergence depends on that of $\sum_{n=0}^{\infty} \frac{4^n}{n^{3/2}\sqrt{\pi}} K^n \|u\|^n$, which only holds by assumption. $\square$

For the generating function $\mathcal{R}[U] = \sum_{n=0}^{\infty} R_n[X, ..., X]$ for matricial cumulants, we also have the following corollary:

**Corollary 2.4.0.8.** *Under the conditions of Proposition 2.4.0.7, with the assumption* $\|u\| \leq \frac{1}{4K}$ *replaced with* $\sup_{k>0} \|u_k\| \leq \frac{1}{4K}$*, the infinite matrix* $\mathcal{R}[U]$ *is bounded, with*

$$\|\mathcal{R}[U]\| \leq \sum_{n=2}^{\infty} \|\phi\| C_{n-2} K^{n-2} \left( \sup_{k>0} \|u_k\| \right)^n . \tag{2.28}$$

*Proof.* Using Corollary 2.3.2.5, the $i$th diagonal of $\mathcal{R}[U]$ has the bound $\|\phi\| C_{n-2} K^{n-2} \left( \sup_{k>0} \|u_k\| \right)^n$, so the convergence of the sum follows from Lemma 2.4.0.5 and the same argument from the proof of Proposition 2.4.0.7. $\square$

## 2.5 Traciality

In this section, we give conditions under which the vacuum state is tracial. We start with an auxiliary result.

**Definition 2.5.0.1.** On $\mathcal{F}_{\text{alg}}(\mathcal{B})$, define an anti-linear involution by the linear extension of

$$S(u_1 \otimes \ldots \otimes u_n) = u_n^* \otimes \ldots \otimes u_1^*.$$

For $b \in \mathcal{B}$, denote $X_r(b) = SX(b^*)S$. Explicitly, for $n \geq 2$,

$$X_r(u_1 \otimes \ldots \otimes u_n) = u_1 \otimes \ldots \otimes u_n \otimes b + u_1 \otimes \ldots \otimes u_{n-1} \otimes \Lambda(b^* \otimes u_n^*)^*$$
$$+ u_1 \otimes \ldots \otimes u_{n-1}(\gamma + \phi)[u_n b],$$

with appropriate modifications for $n = 1$ and $n = 0$. Denote $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi; r) = S\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)S$ the algebra generated by $\{X_r(b) : b \in \mathcal{B}\}$. For $\vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$ a simple tensor, denote $W_r(\vec{\xi}) = SW(\vec{\xi})S$. Then $W_r(S(\vec{\xi}))\Omega = \vec{\xi}$, so that $\Omega$ is cyclic for $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$.

**Proposition 2.5.0.2.**

a. *Suppose that* $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$ *and* $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$ *commute. Then* $W$ *extends to a linear map on* $\mathcal{F}_{alg}(\mathcal{B})$,

$$\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi) = \left\{ W(\vec{\xi}) : \vec{\xi} \in \mathcal{F}_{alg}(\mathcal{B}) \right\}, \quad \Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi; r) = \left\{ W_r(\vec{\xi}) : \vec{\xi} \in \mathcal{F}_{alg}(\mathcal{B}) \right\},$$

*and for* $\vec{\xi} \in \mathcal{F}_{alg}(\mathcal{B})$,

$$W(\vec{\xi})^* = W(S(\vec{\xi})),$$

*Therefore for* $A \in \Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, $S(A\Omega) = A^*\Omega$.

b. *In addition to the assumption in (a), suppose that* $\phi$ *is tracial on* $\mathcal{B}$. *Then the vacuum state is tracial on* $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$.

*Proof.*

a. Since $\Omega$ is cyclic for $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$ and $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi; r)$, it is separating for them (and the von Neumann algebras they generate). Using the recursion and induction, for $\vec{\xi} \in \mathcal{F}_{\text{alg}}(\mathcal{B})$

$$\begin{aligned} W(a^+(b)(\vec{\xi}))\Omega &= X(b)W(\vec{\xi})\Omega - W(a^0(b)(\vec{\xi}))\Omega - W(a^-(b)(\vec{\xi}))\Omega \\ &= X(b)\vec{\xi} - a^0(b)(\vec{\xi}) - a^-(b)(\vec{\xi}) \\ &= a^+(b)(\vec{\xi}). \end{aligned}$$

In particular, if $\vec{\xi} = 0$, then $W(\vec{\xi})\Omega = 0$, so $W(\vec{\xi}) = 0$, and the map $W$ is well defined. Clearly, $W(\vec{\xi}) \in \Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$. On the other hand, using induction on the number of factors, a monomial $X(u_0)X(u_1)\dots X(u_n)$ is a linear combination of terms of the form $X(u_0)W(\vec{\xi})$, each of which is a linear combination of the terms of the form $W(\vec{\xi'})$ by the recursion.

Next, note that

$$X_r(b)\Omega = X(b)\Omega$$

and

$$W(b)^* = X(b)^* = X(b^*) = W(b^*).$$

Using induction,

$$\begin{aligned} W(a^+(b)(\vec{\xi}))^* &= W(\vec{\xi})^* X(b)^* - W(a^0(b)(\vec{\xi}))^* - W(a^-(b)(\vec{\xi}))^* \\ &= W(S(\vec{\xi}))X(b^*) - W(S(a^0(b)(\vec{\xi}))) - W(S(a^-(b)(\vec{\xi}))). \end{aligned}$$

On the other hand, if $X_r(b)$ commutes with $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$,

$$\begin{aligned} W(S(\vec{\xi}))X(b^*)\Omega &= W(S(\vec{\xi}))X_r(b^*)\Omega = X_r(b^*)W(S(\vec{\xi}))\Omega \\ &= SX(b)SS(\vec{\xi}) = SX(b)(\vec{\xi}). \end{aligned}$$

Since $\Omega$ is separating for $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, this implies that

$$W(S(\vec{\xi}))X(b^*) = W(SX(b)(\vec{\xi})) = W(S(a^+(b)(\vec{\xi}))) + W(S(a^0(b)(\vec{\xi}))) + W(S(a^-(b)(\vec{\xi}))).$$

Thus

$$W(a^+(b)(\vec{\xi}))^* = W(S(a^+(b)(\vec{\xi}))).$$

b. We first show that $X_r(b)^*\Omega = X(b)^*\Omega$. Indeed, for $\vec{\xi} \in \mathcal{B}^{\otimes n}$, $\left\langle X_r(b)(\vec{\xi}), \Omega \right\rangle = 0$ if $n \neq 1$.

For $u \in \mathcal{B}$, using the fact that $\phi$ is a trace,

$$\langle X_r(b)(u), \Omega \rangle = \left\langle Sa^-(b^*)(S(u)), \Omega \right\rangle = \overline{\phi[b^*u]} = \phi[ub] = \phi[bu] = \langle u, b^* \rangle = \langle u, X(b)^*\Omega \rangle$$

To check that the vacuum state is tracial on $\Gamma^{alg}_{\gamma,\Lambda}(\mathcal{B}, \phi)$, it suffices to verify that

$$\begin{aligned}
\left\langle W(\vec{\xi})X(b)\Omega, \Omega \right\rangle &= \left\langle W(\vec{\xi})X_r(b)\Omega, \Omega \right\rangle = \left\langle X_r(b)W(\vec{\xi})\Omega, \Omega \right\rangle \\
&= \left\langle W(\vec{\xi})\Omega, X_r(b)^*\Omega \right\rangle = \left\langle W(\vec{\xi})\Omega, X(b)^*\Omega \right\rangle \\
&= \left\langle X(b)W(\vec{\xi})\Omega, \Omega \right\rangle .
\end{aligned}$$

□

**Theorem 2.5.0.3.** *Suppose the operators $X(f) : u \in \mathcal{B}$ are bounded. Denote*

$$\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi) = W^*(X(u) : u \in \mathcal{B}) = W^*(X(u) : u \in \mathcal{B}^{sa}).$$

*Consider the conditions*

$$\Lambda(v^* \otimes u^*)^* = \Lambda(u \otimes v), \tag{2.29}$$

$$u\gamma[yv] - \gamma[uy]v = \Lambda(u \otimes \Lambda(y \otimes v)) - \Lambda(\Lambda(u \otimes y) \otimes v), \tag{2.30}$$

$$\Lambda(u \otimes y\gamma[zv]) - \Lambda(u \otimes y)\gamma[zv] = \Lambda(\gamma[uy]z \otimes v) - \gamma[uy]\Lambda(z \otimes v) \tag{2.31}$$

*and*

$$\phi[\gamma[u]v] = \phi[u\gamma[v]]. \tag{2.32}$$

*Then*

*a. Each $X_r(u)$ commutes with $\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi)$ if and only if conditions* (2.29)*,* (2.30)*,* (2.31)*, and* (2.32) *hold. Note that if $\gamma$ is scalar-valued, the third condition is true automatically.*

*b. If $\phi$ is tracial and each $X_r(u)$ commutes with $\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi)$, then the vacuum state is tracial on $\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi)$.*

*c. If the vacuum state is tracial on $\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi)$, then conditions* (2.29)*,* (2.30)*, and* (2.31) *hold.*

*Under the assumptions in (b), all the conclusions in Proposition 2.5.0.2 hold, the map $S$ extends to an anti-linear isometry on $\mathcal{F}_{\gamma,\phi}(\mathcal{B})$, and the map $A \mapsto SA^*S$ implements the canonical anti-isomorphism between $\Gamma_{\gamma,\Lambda}(\mathcal{B}, \phi)$ and its commutant.*

*Proof.* For part (a), note that since we are working on a Fock space with depth two action, to show that $X(u)X_r(v) = X_r(v)X(u)$ it suffices to consider their actions on tensors of length $0$, $1$, and $2$. A calculation shows that this is equivalent to conditions (2.29)-(2.32) together with the previously obtained (2.2).

For part (c), if the vacuum state is tracial, the joint free cumulants are cyclically symmetric. Using Proposition 2.2.3.2 for cumulants of order up to $5$, we obtain the conditions (2.29)-(2.31).

Part (b), and the rest of the statements, follow from Theorem 2.5.0.2. □

Note that for $\Lambda(u \otimes v) = \Lambda(u)v$, the first condition says $(\Lambda(v)u)^* = \Lambda(u)v$, which is only satisfied in trivial cases.

**Remark 2.5.0.4.** In the setting of Example 2.1.1.1, condition (2.29) translates to

$$\lambda(s, x, y) = \overline{\lambda(s, y, x)},$$

so that $\lambda$ is conjugate-symmetric in all of its arguments. Condition (2.31) holds automatically because $\Lambda$ is self-adjoint and using (2.29). From the final condition,

$$w(s, t) = w(t, s),$$

so if $\gamma$ is scalar-valued, it has to be a multiple of $\phi$.

**Example 2.5.0.5.** It is easy to verify that in Example 2.1.3.1, the vacuum state is always tracial.

### 2.5.1 Further Examples

**Example 2.5.1.1.** Let $\gamma = 0$ and $\Lambda(f \otimes g) = fg$. For any $(\mathcal{B}, \phi)$, the corresponding algebra of operators on a Fock space is the free (compound) Poisson algebra, a particular case of the constructions in the Appendix of [22] (for $q = 0$) or Proposition 23 in [23] (for the scalar case). See also [24].

The following theorem states that in the tracial case, the algebra for $\gamma = 0$ can always be brought into the form of the preceding example. Note that the initial algebra structure of $\mathcal{B}$ plays no part in this result.

**Theorem 2.5.1.2.** *Suppose that $\gamma = 0$, and the vacuum state is tracial. Denote by $\mathcal{H}$ the inner product space $\mathcal{B}$ with the involution and the inner product $\langle f, g \rangle = \phi[g^* f]$ satisfying the relation*

$$\langle f, g \rangle = \langle g^*, f^* \rangle .$$

*Denote*

$$f \cdot g = \Lambda(f \otimes g). \tag{2.33}$$

*Then*

a. *The free cumulants are*

$$R[X(f_1), \ldots, X(f_n)] = \langle f_1, f_2 \cdot \ldots \cdot f_n \rangle .$$

b. *We have an orthogonal decomposition $\mathcal{H} = \mathcal{Z} \oplus \mathcal{P}$ such that for $f \in \mathcal{Z}^{sa}$, $X(f)$ are semicircular, with orthogonal $f$ corresponding to free $X(f)$, and are also free from $X(g)$ with $g \in \mathcal{P}$.*

c. *Suppose in addition that for some $M$, and all $f, g \in \mathcal{H}$,*

$$\langle \Lambda(f \otimes g), \Lambda(f \otimes g) \rangle \leq M \left\| f \right\|_{\mathcal{B}}^2 \langle g, g \rangle .$$

*Then for $f \in \mathcal{P}^{sa}$, $X(f)$ has a centered free compound Poisson distribution, and in particular a centered free Poisson distribution if $f$ is a projection. Moreover, if $f \cdot g = 0$ then $X(f)$ and $X(g)$ are free.*

*Proof.* The notation (2.33) is meant to suggest that we consider the binary operation $\Lambda$ as a multiplication on $\mathcal{H}$ (different from a multiplication it may have inherited from $\mathcal{B}$). Indeed, equations (2.29), (2.30), and Proposition 2.1.0.5 say that

$$(f \cdot g)^* = g^* \cdot f^*, \quad (f \cdot g) \cdot h = f \cdot (g \cdot h), \quad \langle f \cdot g, h \rangle = \langle g, f^* \cdot h \rangle .$$

That is, $(\mathcal{H}, \cdot, *)$ is an associative star-algebra (linearity/distributivity is immediate), which is represented on $\mathcal{H}$ by left multiplication operators. We immediately get

$$R'[X(f_1), \ldots, X(f_{n-1})] f_n = f_1 \cdot f_2 \cdot \ldots \cdot f_n$$

and so part (a).

Next, denote

$$\mathcal{Z} = \{f \in \mathcal{H} : \forall g \in \mathcal{H}, g \cdot f = 0\}$$

and

$$\mathcal{P} = \operatorname{Span}(\{f \cdot g : f, g \in \mathcal{H}\}).$$

Clearly both of them are subspaces (and in fact ideals). Also, using the properties above, one sees that $\mathcal{Z}$ and $\mathcal{P}$ are orthogonal complements of each other, and $\mathcal{H} = \mathcal{Z} \oplus \mathcal{P}$. Moreover, clearly

$$\mathcal{Z}^* = \{f \in \mathcal{H} : \forall g \in \mathcal{H}, f \cdot g = 0\},$$

and again using the properties above one sees that $(\mathcal{Z}^*)^* = \mathcal{Z}^*$, in other words that $\mathcal{Z}$ is self-adjoint. Clearly $\mathcal{P}$ is self-adjoint. Finally, by the uniqueness of the direct sum decomposition it follows that if $f = z + p$ is self-adjoint, with $z \in \mathcal{Z}$ and $p \in \mathcal{P}$, then $z$ and $p$ are also self-adjoint.

For $f_1, \ldots, f_n \in \mathcal{H}^{sa}$ with at least one of them in $\mathcal{Z}$,

$$R[X(f_1), \ldots, X(f_n)] = \delta_{n=2} \langle f_1, f_2 \rangle$$

and is zero unless both $f_1, f_2 \in \mathcal{Z}$. Part (b) follows.

Under the boundedness assumption in part (c), $\mathcal{P}$ satisfies all the axioms of a Hilbert algebra (Chapters 5 and 6 of [25]). Then we can complete $\mathcal{P}$ to a von Neumann algebra represented on $\mathcal{H}$ (since it acts as zero on $\mathcal{Z}$, it is also represented on $\mathcal{P}$) by left multiplication. Moreover, we have a semi-finite trace $\psi$ on this von Neumann algebra such that $\langle f, g \rangle = \psi[g^* \cdot f]$. So in this case for self-adjoint elements,

$$R[X(f_1), \ldots, X(f_n)] = \psi[f_1 \cdot f_2 \cdot \ldots \cdot f_n].$$

Part (c) now follows from the results cited in the preceding example. □

**Proposition 2.5.1.3.** *Let $\eta, \lambda$ be central elements in $\mathcal{B}$, and $V$ be a $*$-linear map on $\mathcal{B}$, unitary with respect to the inner product coming from $\phi$. Then for $\gamma[f] = \eta f$ and*

$$\Lambda(f \otimes g) = V^{-1}[(Vf)\lambda(Vg)],$$

*the vacuum state on the corresponding algebra is tracial.*

*Proof.*

$$\Lambda(f \otimes g)^* = (V^{-1}[(Vf)\lambda(Vg)])^* = V^{-1}[(Vg^*)\lambda(Vf^*)] = \Lambda(g^* \otimes f^*),$$

$$\begin{aligned} f\gamma[hg] - \gamma[fh]g &= f\eta hg - \eta fhg = 0 \\ &= V^{-1}[(Vf)\lambda(VV^{-1}[(Vh)\lambda(Vg)])] - V^{-1}[(VV^{-1}[(Vf)\lambda(Vh)])\lambda(Vg)] \\ &= \Lambda(f \otimes \Lambda(h \otimes g)) - \Lambda(\Lambda(f \otimes h) \otimes g), \end{aligned}$$

$$
\begin{aligned}
\Lambda(f \otimes h\gamma[kg]) - \Lambda(f \otimes h)\gamma[kg] &= V^{-1}[(Vf)\lambda(Vh\eta kg)] - V^{-1}[(Vf)\lambda(Vh)](V^{-1}V)(\eta kg) \\
&= 0 = V^{-1}[(V\eta fhk)\lambda(Vg)] - (V^{-1}V)(\eta fh)V^{-1}[(Vk)\lambda(Vg)] \\
&= \Lambda(\gamma[fh]k \otimes g) - \gamma[fh]\Lambda(k \otimes g),
\end{aligned}
$$

and the last condition from Theorem 2.5.0.3 follows immediately from centrality of $\eta$. $\square$

In the finite-dimensional, commuting setting of [1], all examples of tracial algebras for such $\gamma$ are of this form. It is easy to see, for example by considering group algebras, that in the non-commutative setting that is not the case. The following proposition provides a general description of such algebras.

**Proposition 2.5.1.4.** *Assume that $\mathcal{B}$ is unital and the vacuum state is tracial.*

*a. If $\Lambda(f \otimes g) = \lambda fg$ for $\lambda$ central, then $\gamma[f] = \eta f$ for $\eta$ central, and we are in the setting of Example 2.1.3.1.*

*b. Suppose $\gamma[f] = \eta f$ for $\eta$ central. Then we have an orthogonal decomposition*

$$
\mathcal{H} = (\mathcal{B}, \langle \cdot, \cdot \rangle_\phi) = \mathcal{N} \oplus \mathcal{N}^\perp = \mathcal{Z} \oplus \mathcal{P} \oplus \mathcal{N}^\perp,
$$

*so that on $\mathcal{Z}$ and $\mathcal{P}$, the behavior is as in the preceding theorem. On $\mathcal{N}^\perp$, $\Lambda(f \otimes g) = \lambda fg$ for $\lambda$ central, and we are in the setting of Example 2.1.3.1. If $f \in \mathcal{N}^\perp$ and either $g \in \mathcal{N}$ or $g \in \mathcal{N}^\perp$ with $fg = gf = 0$, $X(f)$ and $X(g)$ are free.*

*Proof.* Assuming condition (a) and the traciality of the vacuum state, by (2.30), $f\gamma[hg] - \gamma[fh]g = 0$. So $\gamma[g] = \gamma[1]g = g\gamma[1]$, and condition (b) follows.

Assuming condition (b), second condition in Proposition 2.1.0.5 reads

$$
\eta g^* \Lambda(b \otimes f) = \eta \Lambda(b^* \otimes g)^* f = \eta \Lambda(g^* \otimes b) f.
$$

Therefore $\eta \Lambda(b \otimes f) = \eta \lambda b f$, where $\lambda = \Lambda(1 \otimes 1)$.

Let

$$\mathcal{N} = \{f \in \mathcal{B} : \eta f = 0\}.$$

Then $\mathcal{N}$ is a subspace and an ideal, and on it $\gamma = 0$, so the arguments on the theorem above apply. Restricted to $\mathcal{N}^{\perp}$, multiplication by $\eta$ is injective (although not necessarily surjective). Since by (2.29), $\eta\lambda g^* b = \eta g^* b\lambda^*$, it follows that $\lambda$ is, on this subspace, self-adjoint and central, and condition (a) on this subspace follows.

Under assumption (b), all the results of the preceding theorem which do not involve free cumulants or freeness still hold. Note that $\mathcal{N}$ is also an ideal with respect to $\cdot$. For $f_1, \ldots, f_n \in \mathcal{N} \cup \mathcal{N}^{\perp}$ with at least one element from $\mathcal{N}$,

$$R'[X(f_1), \ldots, X(f_n)] = f_1 \cdot f_2 \cdot \ldots \cdot f_n$$

as before, and

$$R[X(f_1), \ldots, X(f_n)] = \langle f_1, f_2 \cdot \ldots \cdot f_n \rangle.$$

Note that since the vacuum state is tracial, the free cumulants are cyclically symmetric. So if

$$f_1, \ldots, f_n \in \mathcal{N} \cup \mathcal{N}^{\perp}$$

with at least one element from each of $\mathcal{N}, \mathcal{N}^{\perp}$, we may assume that $f_1 \in \mathcal{N}^{\perp}$ and $f_2 \cdot \ldots \cdot f_n \in \mathcal{N}$. Then their inner product is zero, which implies freeness. $\square$

# 3. The Secondary Framework

## 3.1 The Secondary Construction

For the second phase, we've identified a specific sub-class of operators which holds many desirable properties, such as a free convolution relationship between the distributions of the $X(f)$ with respect to the vacuum state and those of the underlying $f$ with respect to $\phi$. More details will be given in the next section.

For the purposes of allowing non-unital algebras, we opted to start with a $*$-algebra with a linear functional on it (if unital or a C*-algebra, we can require $\phi$ to be a state), but construct the Fock space over the Hilbert space obtained through the GNS construction.

**Construction 3.1.0.1.** Let $(\mathcal{B}, \phi)$ be a noncommutative $*$-probability space, so that $\mathcal{B}$ is a $*$-algebra, and $\phi$ is a (not necessarily faithful) state on it. Using the GNS construction for $\mathcal{B}$, we may construct a pointed Hilbert space $\mathcal{H}$ (that is, a Hilbert space with a unit vector $\Omega$) such that

- $\mathcal{B}$ acts on $\mathcal{H}$ by densely defined, possibly unbounded operators (whose domains contain $\Omega$).
- $\Omega$ is cyclic for $\mathcal{B}$, and $\phi = \langle \cdot \Omega, \Omega \rangle$.

Throughout, we will assume that the representation of $\mathcal{B}$ on $\mathcal{H}$ is faithful. Denote by $\mathcal{H}^\circ$ the orthogonal complement in $\mathcal{H}$ of $\mathbb{C}\Omega$. Note that for any $f \in \mathcal{B}$, $f\Omega - \phi[f]\Omega \in \mathcal{H}^\circ$. Denote

$$\mathcal{B}^\circ = \{g \in \mathcal{B} : \phi[g] = 0\},$$

so that $\mathcal{B} = \mathcal{B}^\circ \oplus \mathbb{C}$. Finally denote $\Lambda(f, g) = fg - \phi[fg]$ a (not necessarily bounded) operator on $\mathcal{H}$.

Fix $t \geq 0$. On the Fock space $\mathcal{F}(\mathcal{H}^\circ, \Omega; t) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} (\mathcal{H}^\circ)^{\otimes n}$, define the inner product by the linear extension of

$$\langle \xi_1 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \ldots \otimes \eta_n \rangle_t = \delta_{n=k} (t+1) t^{n-1} \prod_{i=1}^{n} \langle \xi_i, \eta_i \rangle .$$

For $f \in \mathcal{B}$, define

$$a^+(f)(\xi_1 \otimes \ldots \otimes \xi_n) = (f\Omega - \phi[f]\Omega) \otimes \xi_1 \otimes \ldots \otimes \xi_n, \quad a^+(f)(\Omega) = (f\Omega - \phi[f]\Omega),$$

$$a^-_\phi(f)(\xi_1 \otimes \ldots \otimes \xi_n) = \langle f\xi_1, \Omega\rangle \, \xi_2 \otimes \ldots \otimes \xi_n,$$

$$a^-(f) = \begin{cases} ta^-_\phi(f) & \text{on } (\mathcal{H}^\circ)^{\otimes n}, n \geq 2, \\ (1+t)a^-_\phi(f) & \text{on } (\mathcal{H}^\circ)^{\otimes n}, n = 1, \\ 0 & \text{on } (\mathcal{H}^\circ)^{\otimes n}, n = 0, \end{cases}$$

$$a^0(f)(\xi_1 \otimes \ldots \otimes \xi_n) = (f\xi_1 - \langle f\xi_1, \Omega\rangle \, \Omega) \otimes \xi_2 \otimes \ldots \otimes \xi_n, \quad a^0(f)\Omega = \phi[f]\Omega.$$

**Lemma 3.1.0.2.** *For each $t \geq 0$ and $f \in \mathcal{B}$, denote*

$$X(f,t) = a^+(f) + a^-(f) + a^0(f) + t\phi[f]$$

*Then*

*a.* $X(f,t) = X(f - \phi[f], t) + (1+t)\phi[f]$.

*b. If $\mathcal{B}$ is a $C^*$-algebra, then, denoting by $\|\cdot\|_t$ the operator norm on $\mathcal{F}(\mathcal{H}^\circ, \Omega; t)$, we have $\|X(f,t)\|_t \leq (1 + 2\sqrt{1+t} + t) \, \|f\|_\mathcal{B}$.*

*Proof.* The proof of part (a) is straightforward.

For part (b), we first apply Proposition 2.4.0.3 with the linear functional $(1+t)\phi$ and the map $\gamma = -\phi$, which gives

$$\left\|a^+(f)\right\|_t = \left\|a^-(f)\right\|_t \leq \sqrt{1+t} \, \|f\|_\mathcal{B} \, .$$

Next, by the Pythagorean theorem, it suffices to check the behavior of $a^0(f)$ on each $(\mathcal{H}^\circ)^{\otimes n}$. For $n = 0$, $\|a^0(f)\Omega\| \leq \|f\|_\mathcal{B}$.

For $n = 1$ and $g\Omega \in \mathcal{H}^\circ$, since the second expression is the projection of $fg\Omega$ onto $\mathcal{H}^\circ$, we have

$$\left\|a^0(f)\right\|_t = \|fg\Omega - \langle fg\Omega, \Omega\rangle \Omega\|_t \leq \|fg\Omega\|_t$$
$$= \sqrt{(1+t)\langle fg\Omega, fg\Omega\rangle} \leq \|f\|_{\mathcal{B}} \|g\Omega\|_t .$$

For $n \geq 2$, notice that the deformed inner product on $H^{\otimes n}$ is just the canonical tensor inner product times $(1+t)t^{n-1}$. Since $a^0(f) = a^0_1(f) \otimes I^{\otimes(n-1)}$, where $a^0_1(f) = a^0(f)$ restricted to $\mathcal{H}$, we have $\|a^0(f)\|_t \leq \|f\|_{\mathcal{B}} \|g\Omega\|_t \cdot 1^{n-1} = \|f\|_{\mathcal{B}} \|g\Omega\|_t$.

Finally, it is clear that $\|\phi[f]tI\|_t \leq t\,\|f\|_{\mathcal{B}}$. Apply the triangle inequality to $X(f,t)$. $\square$

**Notation 3.1.0.3.** Denote

$$\Gamma_a(\mathcal{B}, \phi; t) = \mathrm{Alg}(X(f,t) : f \in \mathcal{B}) = \mathrm{Alg}(X(f,t) : f \in \mathcal{B}^{sa}).$$

Note that it is also equal to the algebra generated by $\{X(f,t) : f \in \mathcal{B}^\circ\}$ and $1$. Denote by

$$\Gamma_w(\mathcal{B}, \phi; t) = W^*(X(f,t) : f \in \mathcal{B})$$

the corresponding von Neumann algebra. The vacuum state $\Phi = \langle \cdot\Omega, \Omega\rangle$ is a state on each of these algebras.

**Remark 3.1.0.4.** Except for $\mathcal{B}^\circ$ not necessarily being an algebra, and $\phi$ not necessarily being faithful, the construction above fits into the primary framework, with the linear functional $(1+t)\phi$, the map $\gamma = -\phi$, and $\Lambda(f \otimes g) = fg - \phi[fg]$. For $f, f_j \in \mathcal{B}^\circ$, the following are the same results from before, but translated and simplified to this specific setting:

- (Lemma 2.1.0.3) The inner product is non-degenerate for $t > 0$.
- (Proposition 2.1.0.5) Each $X(f^*, t) = X(f,t)^*$ in the natural sense.

- (Lemma 2.2.2.2) The Boolean cumulants are

$$B^{\Phi}[X(f_1,t),...,X(f_n,t)] = (1+t) \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} t^{|\pi|-1} \prod_{V \in \pi} \Phi[W(V)], \tag{3.1}$$

where for $V = \left\{f_{i(1)}, f_{i(2)}, \ldots, f_{i(k)}\right\}$ with $i(1) < \ldots < i(k)$ and $k \geq 2$,

$$\begin{aligned}\Phi[W(V)] &= \left\langle a_{\phi}^{-}(f_{i(1)}) a^0(f_{i(2)}) \ldots a^0(f_{i(k-1)}) a^+(f_{i(k)})\Omega, \Omega \right\rangle \\ &= \left\langle f_{i(1)} \Lambda(f_{i(2)}, \Lambda(f_{i(3)}, \ldots, \Lambda(f_{i(k-1)}, f_{i(k)})))\Omega, \Omega \right\rangle,\end{aligned}$$

and for $k = 2$, $\Phi[W(V)] = \langle f_{i(1)} f_{i(2)} \Omega, \Omega \rangle$.

- (Proposition 2.2.3.2) The free cumulants are

$$R^{\Phi}[X(f_1,t),...,X(f_n,t)] = (1+t) \sum_{\pi \in \widetilde{\mathrm{NC}}_{ns}(n)} (-1)^{|\pi|-1} \prod_{V \in \pi} \Phi[W(V)].$$

- (Proposition 2.5.0.2, Theorem 2.5.0.3) $\Omega$ is cyclic and separating for $\Gamma_w(\mathcal{B}, \phi; t)$. Denote $S : \mathcal{F}(\mathcal{H}^\circ, \Omega; t) \to \mathcal{F}(\mathcal{H}^\circ, \Omega; t)$, $S(f_1 \otimes \ldots \otimes f_n) = f_n^* \otimes \ldots \otimes f_1^*$ and $X_r(f,t) = SX(f^*,t)S$. Then

$$W(f_1 \otimes \ldots \otimes f_n)^* = W(f_n^* \otimes \ldots \otimes f_1^*),$$

each $X_r(f,t)$ commutes with $\Gamma_w(\mathcal{B}, \phi; t)$, and $\Omega$ is cyclic for the algebra generated by $\{X_r(f,t) : f \in \mathcal{B}^\circ\}$. If $\phi$ is tracial, then $\Phi$ is tracial, and $S$ is an isometry.

Next we consider the case $t = 0$. Compare with Proposition 4.8 in [14], Proposition A.9 in [26], Lemma 3.9 in [9], Lemma 7.2 and Theorem 7.8 in [27].

**Proposition 3.1.0.5.** *The non-commutative star-probability space $(\Gamma_a(\mathcal{B}, \phi; 0), \Phi)$ is isomorphic to $(\mathcal{B}, \phi)$. If $\mathcal{B}$ is a $C^*$-algebra, then $\Gamma_a(\mathcal{B}, \phi; 0) = \Gamma_c(\mathcal{B}, \phi; 0)$, and if $\mathcal{B}$ is a von Neumann algebra, then $\Gamma_a(\mathcal{B}, \phi; 0) = \Gamma_w(\mathcal{B}, \phi; 0)$.*

*Proof.* If $t = 0$, the Fock space is simply $\mathbb{C}\Omega \oplus \mathcal{H}^\circ \simeq \mathcal{H}$. A short calculation shows that for $\xi \in \mathcal{H}$,

$X(f)\xi = f\xi$, and $\langle X(f)\Omega, \Omega \rangle = \phi[f]$. The remaining claims follow. □

**Corollary 3.1.0.6.** *Let $(\mathcal{B}, \phi)$ be a noncommutative probability space, represented on the pointed Hilbert space $(\mathcal{H}, \Omega)$ with $\Omega$ cyclic and implementing $\phi$. Let $f_1, \ldots, f_n \in \mathcal{B}^\circ$. Denote*

$$\Lambda(f, g) = fg - \phi[fg] \in \mathbb{L}(\mathcal{H}).$$

*Then we have the formula for the Boolean cumulants*

$$B[f_1, \ldots, f_n] = \langle f_1 \Lambda(f_2, \Lambda(f_3, \ldots, \Lambda(f_{n-1}, f_n)))\Omega, \Omega \rangle .$$

*Proof.* Apply equation (3.1) with $t = 0$. □

**Construction 3.1.0.7.** Let $\mathcal{B}, \phi, \mathcal{H}$ be as in Construction 3.1.0.1. On $\mathcal{F}(\mathcal{H}^\circ, \Omega; t)$, define the simpler inner product

$$\langle \xi_1 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \ldots \otimes \eta_n \rangle_t = \delta_{n=k} t^n \prod_{i=1}^{n} \langle \xi_i, \eta_i \rangle .$$

For $f \in \mathcal{B}$, define $a^+(f)$ and $a^0(f)$ as in Construction 3.1.0.1, and $a^-(f) = t a_\phi^-(f)$. Denote

$$Y(f, t) = a^+(f) + a^-(f) + a^0(f) + t\phi[f]$$

as before, and let $\Psi$ be the corresponding vacuum state.

This construction also fits into the framework of the primary phase, with the linear functional $t\phi$ and the map $\gamma = 0$. Therefore, as in Remark 3.1.0.4,

- The inner product $\langle \cdot, \cdot \rangle_t$ is non-degenerate for $t > 0$.
- Each $Y(f^*, t) = Y(f, t)^*$ in the natural sense.

- For $f_j \in \mathcal{B}^\circ$, the free cumulants are $R^\Psi[Y(f_1,t), Y(f_2,t)] = t\phi[f_1 f_2]$ and

$$R^\Psi[Y(f_1,t),\ldots,Y(f_n,t)] = t\,\langle f_1 \Lambda(f_2, \Lambda(f_3,\ldots,\Lambda(f_{n-1},f_n)))\Omega,\Omega\rangle \tag{3.2}$$

$$R^\Psi[Y(f_1,t),Y(f_2,t)] = t\langle f_1 f_2 \Omega, \Omega\rangle.$$

- Unless $\dim \mathcal{B}^\circ \leq 1$ or $\phi$ is a homomorphism, $\Psi$ is not tracial. Indeed, applying Theorem 37 from [19] with $\gamma = 0$ and $\Lambda(f \otimes g) = fg - \phi[fg]$, the vacuum state is tracial only if $\phi$ is tracial and for any $f, g, h \in \mathcal{B}^\circ$, $\phi[fg]h - f\phi[gh] = 0$. Suppose $\dim \mathcal{B}^\circ \geq 2$. Let $f \in \mathcal{B}^\circ$, and $h \in \mathcal{B}^\circ$ linearly independent from it. Then for any $g \in \mathcal{B}^\circ$, $\phi[fg] = 0$, in other words $\mathcal{B}^\circ$ is a subalgebra. This implies that $\phi$ is a homomorphism.

We would like $\Phi$ and $\Psi$ to be defined on the same algebra. Thus let

$$\mathcal{T}(\mathcal{B},\phi) = \mathcal{T}(\mathcal{B}^\circ) = \mathbb{C}1 \oplus \bigoplus (\mathcal{B}^\circ)^{\otimes n}$$

be the tensor algebra of $\mathcal{B}^\circ$ (the first notation keeps track of the dependence on $\phi$). Then the maps

$$\pi_{\Phi,t} : f_1 \otimes \ldots \otimes f_n \mapsto X(f_1,t)X(f_2,t)\ldots X(f_n,t)$$
$$\pi_{\Psi,t} : f_1 \otimes \ldots \otimes f_n \mapsto Y(f_1,t)Y(f_2,t)\ldots Y(f_n,t)$$

are $*$-representations of $\mathcal{T}(\mathcal{B},\phi)$ on the Fock spaces from Constructions 3.1.0.1 and 3.1.0.7, and map it onto the appropriate versions of $\Gamma_a(\mathcal{B},\phi,t)$. Under our assumptions on $(\mathcal{B},\phi)$, these representations are not necessarily faithful. But we can still pull $\Phi, \Psi$ back to states on $\mathcal{T}(\mathcal{B},\phi)$, denoted by $\Phi_t$, $\Psi_t$. Then $W^*(\mathcal{T}(\mathcal{B}^\circ))$ in its representation on $L^2(\mathcal{T}(\mathcal{B}^\circ),\Phi_t)$ is isomorphic to $\Gamma_w(\mathcal{B},\phi,t)$ from Construction 3.1.0.1, while $W^*(\mathcal{T}(\mathcal{B}^\circ))$ in its representation on $L^2(\mathcal{T}(\mathcal{B}^\circ),\Psi_t)$ is isomorphic to $\Gamma_w(\mathcal{B},\phi,t)$ from Construction 3.1.0.7.

It will be convenient to denote the element $f_1 \otimes \ldots \otimes f_n \in \mathcal{T}(\mathcal{B},\phi)$ by $X(f_1)\ldots X(f_n)$. Note that this notation is consistent with the multiplication and involution on $\mathcal{T}(\mathcal{B},\phi)$. Then, for example, we have the equality of joint distributions

$$\mu^{\Phi}_{X(f_1,t),\ldots X(f_n,t)} = \mu^{\Phi_t}_{X(f_1),\ldots X(f_n)}.$$

**Proposition 3.1.0.8.** *Suppose the representation of $\mathcal{B}$ on $\mathcal{H}$ is faithful. Then*

*a. If $f \in \mathcal{B}^\circ$ and $\pi_\Phi[f]\Omega = 0$, then $f = 0$.*

*b. $\Phi_t$, and therefore the representation $\pi_\Phi$, are faithful.*

*Proof.* For (a), since $\Omega$ is separating for $\Gamma_a(\mathcal{B}, \phi, t)$, $\pi_{\Phi,t}[f] = 0$. Thus in particular, for $\xi \in \mathcal{H}^\circ$,

$$X(f,t)\xi = f\Omega \otimes \xi + (f\xi - \langle f\xi, \Omega\rangle \Omega) + \langle f\xi, \Omega\rangle \Omega = 0.$$

It follows that for any $\xi \in \mathcal{H}$, $f\xi = 0$, and therefore $f = 0$.

For (b), let $\vec{\xi} \in \mathcal{T}(\mathcal{B}^\circ)$, and suppose that

$$\Phi_t[(\vec{\xi})^* \vec{\xi}] = \langle \pi_\Phi[\xi]\Omega, \pi_\Phi[\xi]\Omega\rangle = 0.$$

Denote the component of $\vec{\xi}$ in the highest tensor power by $\sum_{i=1}^N f_1^{(i)} \otimes \ldots \otimes f_n^{(i)}$. It suffices to show that this sum is zero.

$$\begin{aligned}0 = \pi_\Phi[\vec{\xi}]\Omega &= \sum_{i=1}^N X(f_1^{(i)}, t) \ldots X(f_n^{(i)}, t)\Omega + \text{ terms in lower tensor components} \\ &= \sum_{i=1}^N f_1^{(i)}\Omega \otimes \ldots \otimes f_n^{(i)}\Omega + \text{ terms in lower tensor components,}\end{aligned}$$

and so

$$\sum_{i=1}^N f_1^{(i)}\Omega \otimes \ldots \otimes f_n^{(i)}\Omega = 0$$

as an element of $(\mathcal{H}^\circ)^{\otimes n}$. Let $\{g_k\Omega : 1 \le k \le K\}$ be a basis for the subspace

$$\text{Span}\left\{ f_j^{(i)}\Omega : 1 \le i \le N, 1 \le j \le n \right\} \subseteq \mathcal{H}^\circ.$$

In particular, for some coefficients, $f_j^{(i)}\Omega = \sum_{k=1}^{K} c_{j,k}^{(i)} g_k \Omega$. It follows from part (a) that then $f_j^{(i)} = \sum_{k=1}^{K} c_{j,k}^{(i)} g_k$. So if

$$0 = \sum_{i=1}^{N} f_1^{(i)}\Omega \otimes \ldots \otimes f_n^{(i)}\Omega = \sum_{i=1}^{N} \sum_{k:[n]\to[K]} \prod_{j=1}^{n} c_{j,k(j)}^{(i)} g_{k(1)}\Omega \otimes \ldots \otimes g_{k(n)}\Omega,$$

then each $\sum_{i=1}^{N} \prod_{j=1}^{n} c_{j,k(j)}^{(i)} = 0$, and therefore $\sum_{i=1}^{N} f_1^{(i)} \otimes \ldots \otimes f_n^{(i)} = 0$. $\square$

For the second phase, we've adapted the techniques of these next papers. The first paper concerns a Fock space construction for which the von Neumann algebra generated by all $t$-gaussians (a special subclass of $X(f)$ operators) is the entire algebra of bounded operators on the Fock space. The second paper we shall highlight explores the von Neumann algebras generated by a finite family of $t$-gaussians, through both free product and conditionally free product constructions.

#### 3.1.1 Wysoczański (2006) [5]

The author constructs the $t$-free non-commutative Gaussian random variables as follows. Fix $t \geq 0$ and a separable Hilbert space $\mathcal{H}$ as the complexification of a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, and define the Fock space $\mathcal{F}_t(\mathcal{H})$ as the closure of

$$\mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$$

under the deformed inner product

$$\langle x_1 \otimes \ldots \otimes x_n, y_1 \otimes \ldots \otimes y_k \rangle_t := \delta_{n=k} t^{n-1} \prod_{i=1}^{n} \langle x_i, y_i \rangle$$

$$\langle \Omega, \Omega \rangle_t := 1.$$

Note the similarity in deformation to the inner product of our second construction.

For $f \in \mathcal{H}_{\mathbb{R}}$, the creation operator $a^+(f)$ is defined in the same way as the previously discussed constructions, while the annihilation operator is defined by the linear (over $\mathcal{F}_t(\mathcal{H})$) extension of:

$$\begin{aligned} a^-(f)(x_1 \otimes ... \otimes x_n) &:= t\langle x_1, f\rangle x_2 \otimes ... \otimes x_n \\ a^-(f)(x) &:= \langle x, f\rangle \Omega \\ a^-(f)(\Omega) &:= 0. \end{aligned}$$

As usual, the annihilator is the adjoint of the creator, so the operator $X(f,t) = a^+(f) + a^-(f)$ is self-adjoint.

The author shows that the von Neumann algebra $M_t$ generated by $\{X(f,t) \,|\, f \in \mathcal{H}_\mathbb{R}\}$ is just $B(\mathcal{F}_t(\mathcal{H}))$ in three major steps. First, he showed that the orthogonal projection onto $\mathbb{C}\Omega$ is in $M_t$ by choosing a sequence in $M_t$ which converges to it in the strong operator topology (see the next section for details). Next, he showed that $\Omega$ is *cyclic* for $M_t$, that is, the span of the vectors $\left\{\left(\prod_{i=1}^k X(f_i,t)\right)\Omega \middle|\, f_i \in \mathcal{H},\, k \in \mathbb{N}\right\}$ is dense in $\mathcal{F}_t(\mathcal{H})$. Finally, he proves a standard result: if the orthogonal projection onto a cyclic vector for a von Neumann algebra belongs to that algebra, then its commutant is just $\mathbb{C}I$, so the vNA must be $B(\mathcal{F}_t(\mathcal{H}))$. In our construction, if we are able to prove the first two steps, the third will immediately follow.

### 3.1.2 Ricard (2006) [6]

The author begins with the same Fock space construction $\mathcal{F}_t(\mathcal{H})$ and $t$-gaussian operators $X(f,t)$ as Wysoczański, except $\mathcal{H}$ is a finite-dimensional complexification of a real Hilbert space $\mathcal{H}_\mathbb{R}$, and the von Neumann algebra of interest is

$$\Gamma_{t,n} := W^*\{X(f,t) \,|\, f \in \mathcal{H}_\mathbb{R}\} = W^*\{X(e_1,t), ..., X(e_n,t)\},$$

where $\{e_1, ..., e_n\}$ form a basis for $\mathcal{H}_\mathbb{R}$.

He begins with the simple case where $\mathcal{H} = \mathbb{C}$, so the von Neumann algebra $\Gamma_{t,1}$ is generated by only one operator. The author shows that the map $\rho : \Gamma_{t,1} \to \Gamma_{1,1}$ given by $\rho(X(f,t)) = \sqrt{t}X(f,1)$ extends to a normal representation, by showing that the spectral measure of $\sqrt{t}X(f,1)$ is absolutely continuous with respect to that of $X(f,t)$. Moreover, $\Gamma_{t,n}$ contains $n$ representations of $\Gamma_{t,1}$, one for each $X(e_i,t)$, which we will denote by $A_i$.

With this in mind, there are two states given on each $A_i$: the vacuum state of $\Gamma_{t,n}$ restricted to $A_i$ (denoted $\phi$) and the vacuum state on $\Gamma_{1,1}$ pulled back to $A_i$ via the representation (denoted $\psi$). The von Neumann subalgebras $(A_i,\phi,\psi)\subset\Gamma_{t,n}$ are conditionally free with respect to $\phi$.

The main objective is to show that for $n\geq 2$,

$$\Gamma_{t,n}=\begin{cases}\Gamma_{1,n} & \text{if } t\in\left[\frac{n}{n+\sqrt{n}},\frac{n}{n-\sqrt{n}}\right]\\ \Gamma_{1,n}\oplus B(\ell_2) & \text{otherwise.}\end{cases}$$

(It is well known that $\Gamma_{1,n}=L(\mathbb{F}_n)$, the free group factor with $n$ generators. Subsection 1.2.4 has everything one needs in order to show this.)

To begin, the author identifies the orthonormal polynomials for $X(e_i,t)$ with respect to $\phi$. Denoting by $U_n$ the Chebyshev polynomials of the second type of degree $n$, the orthonormal polynomials are

$$\begin{aligned}v_0(X)&=1\\ v_1(X)&=X=\sqrt{t}U_1\left(\frac{X}{2\sqrt{t}}\right)\\ v_n(X)&=\sqrt{t}\left(U_n\left(\frac{X}{2\sqrt{t}}\right)-\left(\frac{1}{t}-1\right)U_{n-2}\left(\frac{X}{2\sqrt{t}}\right)\right)\text{ for } n\geq 2.\end{aligned}$$

The orthonormal polynomials for $\sqrt{t}X(e_i,1)$ are $u_n(X)=U_n\left(\frac{X}{2\sqrt{t}}\right)$. The author uses these to show that for $i(1)\neq ...\neq i(\ell)$ and $\alpha_j\geq 1$,

$$u_{\alpha_1}(X(e_{i(1)},t))...u_{\alpha_{\ell-1}}(X(e_{i(\ell-1)},t))v_{\alpha_\ell}(X(e_{i(\ell)},t))\Omega=\left(e_{i(1)}^{\otimes\alpha_1}\right)\otimes...\otimes\left(e_{i(\ell)}^{\otimes\alpha_\ell}\right).$$

This is used to prove that $(A_i,\phi,\psi)\subset\Gamma_{t,n}$ are conditionally free with respect to $\phi$.

The author then defined a new functional on $B(\mathcal{F}_t(\mathcal{H}))$ by (letting $\alpha = \frac{1}{t} - 1$)

$$\eta = \alpha\left(\left(n + \frac{1}{\alpha}\right) - \left(X(e_1,t)^2 + ... + X(e_n,t)^2\right)\right)\Omega,$$

$$\tilde{\psi}(X) = \langle X\Omega, \eta\rangle,$$

which he shows is equivalent to the vacuum state on $C_{1,n}$ pulled back to $C_{t,n}$ (the C*-algebras generated by the usual free gaussians and the $t$-gaussians, respectively) via the representation discussed earlier. This means that same representation extends to a normal, surjective representation $\Gamma_{t,n} \to \Gamma_{1,n}$, hence $\Gamma_{t,n}$ must have a direct summand isomorphic to $\Gamma_{1,n}$.

Then the author proceeds by cases, first $t \notin \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}}\right]$. First, by showing that the C*-algebra $C_{t,n}$ contains a nonzero compact operator, we know that the von Neumann algebra $\Gamma_{t,n}$ is a direct sum of two algebras: $\Gamma_{1,n}$ and either $B(\mathcal{K})$ for $\mathcal{K}$ finite-dimensional or $B(\ell^2)$. He then shows that, for $1-q$ the central support of the representation $\rho$, $q\Omega$ is nonzero, and uses the above fact regarding the state $\langle X\Omega, \eta\rangle$ to show that $q\Omega$ is in $\ker\left(X(e_1,t)^2 + ... + X(e_n,t)^2 - (n + \frac{1}{\alpha})\right)$, which he also proved is one-dimensional. Hence, $q\Gamma_{t,n}$ contains a one-dimensional projection onto a cyclic vector, so it is isomorphic to $B(\ell_2)$.

For the case $t \in \left(\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}}\right)$, the author first shows that $\phi$ is faithful on $\Gamma_{t,n}$ from which it follows that $\Gamma_{t,n}$ contains no compact operator and $C_{t,n}$ and $C_{1,n}$ are isomorphic. Finally, by showing that the representation $\rho$ is faithful, the same is shown for the corresponding von Neumann algebras.

The case for $t = \frac{n}{n\pm\sqrt{n}}$ is quite different. The author defines a normal linear form on $\Gamma_{1,n}$ by

$$\tilde{\phi}(x) = \left\langle x\Omega, \sum_{k\geq 0} \alpha^k \sum_{\substack{|i|=2k \\ i_{2j+1}=i_{2j+2}}} e_{i(1)} \otimes ... \otimes e_{i(2k)} \right\rangle.$$

Then he shows that the GNS construction of $(\Gamma_{1,n}, \tilde{\phi})$ gives the representation $\rho^{-1}$ (that is, with image $\Gamma_{t,n}$).

Finally, the author discusses an extension of these results, one which we aim to adapt in our construction. For $\mathcal{A}_i = L^\infty(I_i, \mu_i)$ ($I_i$ bounded $\mu_i$-measurable subsets of $\mathbb{R}$), we let $\phi_i$ be the usual integration and $\psi_i$ normal states on $A_i$, so each $\psi_i$ has a density $f_i$ with respect to $\phi_i$. They consider the algebraic conditionally free product $\tilde{\mathcal{A}} = *_{i=1}^n(\mathcal{A}_i, \phi_i, \psi_i)$ and its corresponding von Neumann conditionally free product $\mathcal{A}$. One drawback is that $\phi$ is not faithful on $\mathcal{A}$ in general, not even for $t$-gaussians. The injections $i_i : \mathcal{A}_i \to \mathcal{A}$ are still normal, isometric, and state-preserving. However, there is no conditional expectation $\mathcal{A} \to \mathcal{A}_i$ which is state-preserving.

First, if each density $f_i$ is in either $L^2(\mu_i)$ or $L^\infty(\mu_i)$, the free product state $\psi$ extends to a normal state on $\mathcal{A}$. As a result, if each $f_i$ is in $L^2(\mu_i)$, the conditionally free product von Neumann algebra has a direct summand isomorphic to the usual free product. On the other hand, if the densities are bounded and the distribution of

$$c := 1 + \sum_{i=1}^{n} \big(f_i(1_i) - 1_i\big) \quad (\text{ where } 1_i = \text{ identity function on } I_i)$$

with respect to $\phi$ does not have an atom at 0, then the conditionally free product von Neumann algebra is isomorphic to the usual free product.

## 3.2 Cumulants, Convolutions, and Independence

### 3.2.1 The Distributions as Free Additive Convolution Powers

The first important property of this construction is that the free cumulants of the $X(f)$ can be easily expressed in terms of the free cumulants of the $f$. In fact, much more is true.

**Proposition 3.2.1.1.** *Let $\{f_1, \ldots, f_n\} \subset \mathcal{B}$. Then*

$$R^\Phi[X(f_1, t), \ldots, X(f_n, t)] = (1 + t)R^\phi[f_1, \ldots, f_n].$$

*Therefore we have the relation between joint distributions*

$$\mu^\Phi_{X(f_1,t),\ldots X(f_n,t)} = \mu^{\boxplus(1+t)}_{f_1,\ldots,f_n}.$$

*Proof.* First, by Lemma 3.1.0.2A, the fact that $\mathbb{C}$ is freely independent from all subalgebras of a non-commutative probability space, the fact that free independence implies mixed free cumulants are zero, and linearity, we only need to prove this for $\{f_1, \ldots, f_n\} \subset \mathcal{B}^\circ$.

By Proposition 3.1.0.5, we have $R^\phi[f_1, \ldots, f_n] = R^\Phi[X(f_1, 0), \ldots, X(f_n, 0)]$. From Equation (3.1.0.4), we see that $R^\Phi[X(f_1, t), \ldots, X(f_n, t)] = (1 + t)R^\Phi[X(f_1, 0), \ldots, X(f_n, 0)]$. $\square$

**Corollary 3.2.1.2.** *Let $(\mathcal{B}, \phi)$ be a unital star-probability space. For every $t \geq 0$, there is a unital star-probability space $(\mathcal{B}_t, \phi_t)$ and a star-linear map*

$$X(\cdot, t) : (\mathcal{B}, \phi) \to (\mathcal{B}_t, \phi_t)$$

*such that for any $f_1, \ldots, f_n \in \mathcal{B}$, the joint distribution of $X(f_1, t), \ldots, X(f_n, t)$ in $(\mathcal{B}_t, \phi_t)$ is the $(1 + t)$'th free convolution power of the joint distribution of $f_1, \ldots, f_n$ in $(\mathcal{B}, \phi)$.*

**Remark 3.2.1.3.** Let $(\mathcal{A}, \Phi, \mathcal{C})$ be a $\mathcal{C}$-valued probability space. For every c.p. map $\eta$ on $\mathcal{C}$, one can give a version of Construction 3.1.0.1, with the inner product

$$\begin{aligned} &\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n \rangle_\eta \\ &\qquad = \delta_{n=k}((1 + \eta) \circ \mathbb{E}) \left[ g_n^*(\eta \circ \mathbb{E}) \left[ g_{n-1}^*(\eta \circ \mathbb{E}) \left[ \ldots (\eta \circ \mathbb{E})[g_1^* f_1] \ldots \right] f_{n-1} \right] f_n \right] \end{aligned}$$

on $\mathcal{C}\Omega \oplus \bigoplus_{n=1}^\infty (\mathcal{A}^\circ)^{\otimes n}$ (but not on $\mathcal{C}\Omega \oplus \bigoplus_{n=1}^\infty (\mathcal{A}^\circ)^{\otimes cn}$). Then the corresponding forms of Lemma 3.1.0.2, Proposition 3.1.0.5, and Corollary 3.1.0.6 hold with the same proof. In Proposition 3.2.1.1, the formula

$$R[X(f_1, \eta), \ldots, X(f_n, \eta)] = (1 + \eta)[R[X(f_1, 0), \ldots, X(f_n, 0)]].$$

holds as well. Note however that, according to the standard definition of $\mathcal{C}$-valued distribution, this is not sufficient to claim that the joint distribution of the variables on the left-hand side is the $(1 + \eta)$'th free convolution power of the joint distribution of the variables on the right-hand side.

**Example 3.2.1.4.** If $p_1, \ldots, p_n$ are orthogonal projections adding up to the identity, then so are $X(p_1)$, ..., $X(p_n)$. Therefore the joint distribution of $X(p_1, t), \ldots, X(p_n, t)$ is free multinomial in the sense of Section 4.6 of [1], that is, the joint distribution is the $t$th free additive convolution power of the (classic) multinomial distribution

$$\mu(x) = q_1 \delta_{e_1} + q_2 \delta_{e_2} + \ldots + q_d \delta_{e_d}.$$

### 3.2.2 Conditionally Free Cumulants

**Proposition 3.2.2.1.** *Let* $\{f_1, \ldots, f_n\} \subset \mathcal{B}^\circ$.

a. *The free cumulants with respect to the state* $\Psi_t$ *are*

$$\begin{aligned} R^{\Psi_t}[X(f_1), \ldots, X(f_n)] &= t \left\langle f_1 \Lambda(f_2, \Lambda(f_3, \ldots, \Lambda(f_{n-1}, f_n))) \Omega, \Omega \right\rangle \\ &= t B^\phi[f_1, \ldots, f_n]. \end{aligned}$$

*Therefore we have the relation between joint distributions*

$$\mu^{\Psi_t}_{X(f_1), \ldots X(f_n)} = \mathbb{B}_t(\mu_{f_1, \ldots, f_n})^{\uplus t} = \left( \mu^{\boxplus(1+t)}_{f_1, \ldots, f_n} \right)^{\uplus \frac{t}{1+t}}.$$

b. *The conditionally free cumulants with respect to the pair* $(\Phi, \Psi)$ *are*

$$\begin{aligned} R^{(\Phi_t, \Psi_t)}[X(f_1), \ldots, X(f_n)] &= (1+t) \left\langle f_1 \Lambda(f_2, \Lambda(f_3, \ldots, \Lambda(f_{n-1}, f_n))) \Omega, \Omega \right\rangle \\ &= (1+t) B^\phi[f_1, \ldots, f_n]. \end{aligned}$$

*Proof.* (a) Combine equation (3.2) with Corollary 3.1.0.6. Next, using Proposition 3.5, equation

(6.11), and Definition 4.1 from [28],

$$\begin{aligned} B^{\Psi_t}[X(f_1), \ldots, X(f_n)] &= \sum_{\pi \in \widetilde{NC}(n)} R_\pi^{\Psi_t}[X(f_1), \ldots, X(f_n)] \\ &= \sum_{\pi \in \widetilde{NC}(n)} t^{|\pi|} B_\pi^\phi[f_1, \ldots, f_n] \\ &= t B_{\mathbb{B}_t(\mu_{f_1,\ldots,f_n})} \\ &= B_{\mathbb{B}_t(\mu_{f_1,\ldots,f_n})^{\uplus t}} \\ &= B_{\left(\mu_{f_1,\ldots,f_n}^{\boxplus(1+t)}\right)^{\uplus \frac{t}{1+t}}}. \end{aligned}$$

(b) First, $R^{(\Phi_t,\Psi_t)}[X(f_1)] = 0$. Next,

$$\begin{aligned} \sum_{\pi \in \mathrm{Int}(n)} B^{\Phi_t}[X(f_1), \ldots, X(f_n)] &= \Phi_t[X(f_1) \ldots X(f_n)] \\ &= \sum_{\pi \in \mathrm{NC}(n)} \left( \prod_{V_\ell \in \pi \text{ (inner)}} R^{\Psi_t}[V_\ell] \right) \left( \prod_{V_\ell \in \pi \text{ (outer)}} R^{\Phi_t,\Psi_t}[V_\ell] \right). \end{aligned}$$

The left-hand side is equal to

$$\sum_{\pi \in \mathrm{Int}(n)} \prod_{V \in \pi} \left( (1+t) \sum_{\sigma \in \widetilde{\mathrm{NC}}_{ns}(V)} t^{|\sigma|-1} \prod_{W \in \sigma} B^\phi[W] \right),$$

while the right-hand side, by Part A, is equal to

$$\sum_{\pi \in \mathrm{NC}(n)} \left( \prod_{V_\ell \in \pi \text{ (inner)}} t B^\phi[V_\ell] \right) \left( \prod_{V_\ell \in \pi \text{ (outer)}} R^{\Phi_t,\Psi_t}[V_\ell] \right).$$

From this, the claim inductively follows, since each $\pi \in \mathrm{NC}(n)$ can be uniquely constructed from an interval partition by replacing each block $V$ with some $\sigma \in \widetilde{\mathrm{NC}}(V)$, a partition of the elements of the block. $\square$

**Corollary 3.2.2.2.**

a. *Suppose $\{f_1, \ldots, f_n\}$ are freely independent in $(\mathcal{B}, \phi)$. Then $\{X(f_1), \ldots, X(f_n)\}$ are free in $(\mathcal{T}(\mathcal{B}, \phi), \Phi_t)$.*

b. *Suppose $\{f_1, \ldots, f_n\} \subset \mathcal{B}^\circ$ are Boolean independent in $(\mathcal{B}, \phi)$. Then $\{X(f_1), \ldots, X(f_n)\}$ are freely independent in $(\mathcal{T}(\mathcal{B}, \phi), \Psi_t)$ and conditionally free in $(\mathcal{T}(\mathcal{B}, \phi), \Phi_t, \Psi_t)$.*

*Proof.* Using the characterizations of free, Boolean, and conditionally free independence in terms of their respective cumulants, Part A follows from Proposition 3.2.1.1, while Part B follows from Proposition 3.2.2.1. □

### 3.3 von Neumann Algebras

We now have a more thorough understanding of the distributions of $X(f, t)$. In this section, we proceed to investigate the von Neumann algebras generated by $\{X(f_i, t)\}_{i=1}^n$ for a fixed $t > 0$ for several examples of *-algebras $\mathcal{B}$ and generating sets $\{f_i\} \subset (\mathcal{B}^\circ)^{sa}$.

#### 3.3.1 $\mathcal{B} = \mathbb{C}^2$

In this section, fix $\alpha \in (0, 1)$. Let $p \in M_2(\mathbb{C})$ be a projection such that $p_{11} = \alpha$. The algebra $\mathcal{B}$ generated by $p$ and $1$ is isomorphic to $\mathbb{C}^2$, and the vector $(1, 0)$ is cyclic for it. With $\phi$ the vector state of $(1, 0)$, $\mathcal{B}^\circ$ is spanned by $p^\circ = p - \alpha$, and $\mathcal{B} \simeq \mathbb{C}$ is also generated by $p^\circ$ and $1$. So $\Gamma(\mathcal{B})$ is generated by a single element $X(p^\circ)$ and $1$. The explicit forms of $p$ and $p^\circ$ are the matrices

$$p = \begin{pmatrix} \alpha & \sqrt{\alpha(1-\alpha)} \\ \sqrt{\alpha(1-\alpha)} & 1-\alpha \end{pmatrix}, \quad \text{and} \quad p^\circ = \begin{pmatrix} 0 & \sqrt{\alpha(1-\alpha)} \\ \sqrt{\alpha(1-\alpha)} & 1-2\alpha \end{pmatrix}.$$

Without loss of generality, we may assume that $\alpha \leq \frac{1}{2}$.

**Theorem 3.3.1.1.** a. *The distribution of $p$ is Bernoulli with parameter $\alpha$.*

b. *The distribution of $X(p^\circ, t)$ is*

$$d\mu^{\Phi}_{X(p^\circ,t)} = \frac{(t+1)\sqrt{4t\alpha(1-\alpha) - \big((1-2\alpha) - x\big)^2}}{2\pi(\alpha(1+t) + x)((1-\alpha)(1+t) - x)} dx, \tag{3.3}$$

*with support* $\left[1-2\alpha-2\sqrt{t\alpha(1-\alpha)}, 1-2\alpha+2\sqrt{t\alpha(1-\alpha)}\right]$ *and atoms*

$$\mu^{\Phi}_{X(p^{\circ},t)}(\{-\alpha(1+t)\}) = \max\{1-\alpha(1+t), 0\}$$
$$\mu^{\Phi}_{X(p^{\circ},t)}(\{(1-\alpha)(1+t)\}) = \max\{\alpha(1+t)-t, 0\}$$

*In particular, for* $\alpha = \frac{1}{2}$,

$$d\mu^{\Phi}_{X(p^{\circ},t)} = \frac{(t+1)\sqrt{t-x^2}}{2\pi\left(\left(\frac{1+t}{2}\right)^2 - x^2\right)}\,dx, \tag{3.4}$$

*and*

$$\mu^{\Phi}_{X(p^{\circ},t)}\left(\left\{\pm\frac{1+t}{2}\right\}\right) = \max\left\{\frac{1-t}{2}, 0\right\}.$$

c. *The* $W^*$*-probability space* $(\Gamma_w(\mathcal{B},\phi;t),\Phi)$ *is isomorphic to* $L^{\infty}(\mu^{\Phi}_{X(p^{\circ},t)}) \oplus \mathbb{C} \oplus \mathbb{C}$.

d. *The distribution of* $X(p^{\circ},t)$ *with respect to* $\Psi$ *is*

$$d\mu^{\Psi}_{X(p^{\circ},t)} = \frac{\sqrt{4t\alpha(1-\alpha) - \left((1-2\alpha)-x\right)^2}}{2\pi(t\alpha(1-\alpha)+(1-2\alpha)x)}dx, \tag{3.5}$$

*with support* $\left[1-2\alpha-2\sqrt{t\alpha(1-\alpha)}, 1-2\alpha+2\sqrt{t\alpha(1-\alpha)}\right]$ *and, for* $\alpha \neq \frac{1}{2}$, *an atom*

$$d\mu^{\Psi}_{X(p^{\circ},t)}\left(\left\{\frac{-t\alpha(1-\alpha)}{1-2\alpha}\right\}\right) = \max\left\{1 - t\frac{\alpha(1-\alpha)}{(1-2\alpha)^2}, 0\right\}$$

*For* $\alpha = \frac{1}{2}$,

$$d\mu^{\Psi}_{X(p^{\circ},t)} = \frac{2}{\pi t}\sqrt{t-x^2}\,dx. \tag{3.6}$$

*Proof.* (a) follows from direct computation and observing the moments are $1, \alpha, \alpha, \alpha, \ldots$.

For (b). we will use 3.2.1.1. The Cauchy transform of the distribution of $p$ is

$$G(z) = \frac{1-\alpha}{z} + \frac{\alpha}{z-1},$$

whose inverse is

$$z = \frac{\eta + 1 \pm \sqrt{(\eta+1)^2 - 4\eta(1-\alpha)}}{2\eta}.$$

Subtracting $\frac{2}{2\eta}$ from this gives the R-transform. To apply Proposition 3.2.1.1 requires the distribution of $p - \alpha$, whose R-transform is

$$z = \frac{(1-2\alpha)\eta - 1 \pm \sqrt{(\eta+1)^2 - 4\eta(1-\alpha)}}{2\eta}.$$

The R-transform of its $(1+t)$th convolution power is

$$z = (1+t)\frac{(1-2\alpha)\eta - 1 \pm \sqrt{(\eta+1)^2 - 4\eta(1-\alpha)}}{2\eta}.$$

Adding $\frac{2}{2\eta}$ to this and inverting gives the corresponding Cauchy transform

$$G_{1+t}(z) = \frac{(1-2\alpha)(1+t) - (1-t)z - (t+1)\sqrt{4\alpha^2(1+t) - 4\alpha(1+t-z) + (z-1)^2}}{2(\alpha(1+t)+z)((1-\alpha)(1+t)-z)}.$$

Applying Stieltjes inversion gives the result.

Applying Proposition 8 from Chapter 3 of [15] gives the atoms.

(d) Since $p$ must have the form $\begin{pmatrix} \alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}$ where $\beta^2 = \alpha(1-\alpha)$, $p^\circ = \begin{pmatrix} 0 & \beta \\ \beta & 1-2\alpha \end{pmatrix}$, so by 3.2.2.1,

$$\begin{aligned} R_n^\Psi[X(p^\circ, t)] &= tB_n^\phi[p^\circ] \\ &= t\langle p^\circ \Lambda(p^\circ, \Lambda(p^\circ, ..., \Lambda(p^\circ, p^\circ)...))\Omega, \Omega\rangle \\ &= t\alpha(1-\alpha)(1-2\alpha)^{n-2}, \end{aligned}$$

and so the R-transform with respect to $\Psi$ is the geometric series

$$t\alpha(1-\alpha)\sum_{n=1}^{\infty}(1-2\alpha)^{n-1}z^n.$$

This can be written in the form

$$\frac{t\alpha(1-\alpha)z}{1-(1-2\alpha)z}.$$

Add $\frac{1}{z}$ to this and invert to get the Cauchy transform

$$z=\frac{1-2\alpha+\eta\pm\sqrt{(1-2\alpha+\eta)^2-4(t\alpha(1-\alpha)+(1-2\alpha)\eta)}}{2(t\alpha(1-\alpha)+(1-2\alpha)\eta)}.$$

Apply Stieltjes inversion to get the absolutely continuous part. We have an atom at $x=\frac{-t\alpha(1-\alpha)}{1-2\alpha}$, with measure $\max\left\{1-t\frac{\alpha(1-\alpha)}{(1-2\alpha)^2},0\right\}$. $\square$

**Corollary 3.3.1.2.** *Fix $\alpha=\frac{1}{2}$. Re-scale the time by $t=\frac{\theta}{1-\theta}$ and the variable itself by $2\sqrt{1-\theta}$, so that we consider*

$$2\sqrt{1-\theta}X(p^\circ).$$

*The distribution of this variable with respect to $\Phi_t$ is*

$$\frac{\sqrt{4\theta-x^2}}{2\pi(1-(1-\theta)x^2)}\,dx+\max\left\{\frac{1-2\theta}{2(1-\theta)},0\right\}(\delta_{-1/\sqrt{1-\theta}}+\delta_{1/\sqrt{1-\theta}}),$$

*and with respect to $\Psi_t$ it is*

$$\frac{1}{2\pi\theta}\sqrt{4\theta-x^2}\,dx.$$

Note that these are exactly the two distributions of the $\theta$-Gaussian element in Proposition 2.1 [6]. So up to this re-scaling, we can apply his results directly to describe the von Neumann algebra.

### 3.3.2 $\mathcal{B}=*_{i=1}^d\mathbb{C}^2$ (Free product of 3.3.1)

**Notation 3.3.2.1.** In this section, we will take $d$ copies of $\mathbb{C}^2$, where the $i$th copy is generated by the identity and the projection with state $\alpha_i$ (assuming without loss of generality again that $\alpha_i\leq\frac{1}{2}$). Borrowing notation from [13], this information is denoted $\mathbb{C}^2=\underset{\alpha_i}{\overset{p_i}{\mathbb{C}}}\oplus\underset{1-\alpha_i}{\overset{1-p_i}{\mathbb{C}}}$.

**Theorem 3.3.2.2.** *Let $\mathcal{B}$ be the free product $*_{i=1}^d\mathbb{C}^2$. Then in $(\Gamma_w(\mathcal{B},\phi;t),\Phi)$, the von Neumann* *subalgebra generated by $X(p_i^\circ)$ is*

$$W^*(X(p_i^\circ) : 1 \le i \le d) \simeq \begin{cases} \underset{1-\gamma_1-\gamma_2}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2}{\mathbb{C}}, & t < \frac{\alpha_d - \sum_{i=1}^{d-1}\alpha_i}{1-\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)}, \\ \underset{1-\gamma_1}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_1}{\mathbb{C}}, & \frac{\alpha_d - \sum_{i=1}^{d-1}\alpha_i}{1-\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)} \le t < \frac{1-\left(\left(\sum_{i=1}^{d}\alpha_i\right)\right)}{\left(\sum_{i=1}^{d}\alpha_i\right)}, \\ \mathbb{F}_d, & \frac{1-\left(\left(\sum_{i=1}^{d}\alpha_i\right)\right)}{\left(\sum_{i=1}^{d}\alpha_i\right)} \le t, \end{cases}$$

*where*

$$\gamma_1 = \max\left\{1 - \left(\sum_{i=1}^{d}\alpha_1\right)(1+t), 0\right\}$$
$$\gamma_2 = \max\left\{\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)(1+t) - t, 0\right\}$$

*and $x$ is chosen so that the free dimension is the sum of the free dimensions of $W^*(X(p_i^\circ))$.*

*Proof.* Since $\alpha_d \le \frac{1}{2}$,

$$\frac{\alpha_d - \sum_{i=1}^{d-1}\alpha_i}{1-\left(\alpha_d - \sum_{i=1}^{d-1}\alpha_i\right)} \le \frac{1-\left(\left(\sum_{i=1}^{d}\alpha_i\right)\right)}{\left(\sum_{i=1}^{d}\alpha_i\right)}.$$

So the statement of the theorem is equivalent to $W^*(X(p_i^\circ) : 1 \le i \le d) \simeq \underset{1-\gamma_1-\gamma_2}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_1}{\mathbb{C}} \oplus \underset{\gamma_2}{\mathbb{C}}$, for $\gamma_1, \gamma_2$ as above. We will prove this by induction on $d$. For $d = 1$, this follows from Theorem 3.3.1.1. Since by assumption, $\{p_i^\circ : 1 \le i \le d\}$ are free in $(\mathcal{B}, \phi)$, by Corollary 3.2.2.2,

$$W^*(X(p_i^\circ) : 1 \le i \le d) = *_{i=1}^{d} W^*(X(p_i^\circ)).$$

Since the free product is a commutative operation, without loss of generality we may assume that $\alpha_i$'s are increasing.

Suppose the statement holds for $d$. Then by Theorem 2.4 of [13] (provided in Subsection 1.2.5.1

for convenience),

$$(\Gamma(\mathcal{B}), \Phi)^{*(d+1)} \simeq \underset{1-\gamma_{11}-\gamma_{12}-\gamma_{21}-\gamma_{22}}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma_{11}}{\mathbb{C}} \oplus \underset{\gamma_{12}}{\mathbb{C}} \oplus \underset{\gamma_{21}}{\mathbb{C}} \oplus \underset{\gamma_{22}}{\mathbb{C}},$$

where

$$\begin{aligned}
\gamma_{11} &= \max\left\{1 - \left(\sum_{i=1}^{d+1} \alpha_i\right)(1+t), 0\right\} \\
\gamma_{12} &= \max\left\{\left(\alpha_d - \alpha_{d+1} - \sum_{i=1}^{d-1} \alpha_i\right)(1+t) - t, 0\right\} \\
\gamma_{21} &= \max\left\{\left(\alpha_{d+1} - \sum_{i=1}^{d} \alpha_i\right)(1+t) - t, 0\right\} \\
\gamma_{22} &= \max\left\{\left(\alpha_{d+1} + \alpha_d - \sum_{i=1}^{d-1} \alpha_i\right)(1+t) - 2t - 1, 0\right\}.
\end{aligned}$$

Note that this expansion holds even if some of $\gamma_1, \gamma_2$ are zero. Since $\alpha_d \leq \alpha_{d+1}$, $\gamma_{12} = 0$. Since $\alpha_d, \alpha_{d+1} \leq \frac{1}{2}$, $\gamma_{22} = 0$. Finally, $\gamma_{11}$ and $\gamma_{21}$ are precisely the forms of $\gamma_1$, $\gamma_2$ for $d+1$. The result follows. $\square$

### 3.3.3 $\mathcal{B} = M_{d+1}(\mathbb{C})$, $d \geq 2$

**Remark 3.3.3.1.** Following [29], for $1 \leq i \leq d$, let $\mathcal{H}_i = \mathbb{C}\Omega_i \oplus \mathcal{H}_i^\circ$ be pointed Hilbert spaces, and $(\mathcal{B}_i, \phi_i)$ be star-probability spaces represented on them as in Construction 3.1.0.1. Let

$$\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{i=1}^{d} \mathcal{H}_i^\circ.$$

Represent each $\mathcal{B}_i$ on $\mathcal{H}$ by

$$\begin{aligned}
b_i(s\Omega \oplus \xi_1 \oplus \ldots \oplus \xi_d) = s\phi_i[b_i]\Omega \oplus 0 \oplus \ldots \oplus (sb_i\Omega - s\phi_i[b_i]\Omega) \oplus \ldots \oplus 0 \\
+ \langle b_i\xi_i, \Omega_i\rangle\, \Omega \oplus 0 \oplus \ldots \oplus (b_i\xi_i - \langle b_i\xi_i, \Omega_i\rangle\, \Omega) \oplus \ldots \oplus 0.
\end{aligned}$$

Let $\mathcal{B}$ be the algebra generated by these non-unital embeddings of $\{\mathcal{B}_i : 1 \leq i \leq d\}$ in $\mathbb{B}(\mathcal{H})$, and $\phi$ the vector state given by $\Omega$. Then $\{\mathcal{B}_i : 1 \leq i \leq d\}$ are Boolean independent in $(\mathcal{B}, \phi)$. Clearly $\phi$ is faithful; if each representation of $\mathcal{B}_i$ on $\mathcal{H}_i$ is faithful, so are their representations on $\mathcal{H}$.

**Corollary 3.3.3.2.** *Let $\{\alpha_i : 1 \leq i \leq d\} \subset (0,1)$. Consider the algebra $M_{d+1}(\mathbb{C})$ with the (not faithful) vector state $\phi_{11}$ given by the $(1,1)$ entry. In this algebra, we can choose projections $\{p_i : 1 \leq i \leq d\}$ such that $\{p_i^\circ = p_i - \alpha_i : 1 \leq i \leq d\}$ are centered, Boolean independent, and generate $M_{d+1}(\mathbb{C})$.*

*Proof.* Let $\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{i=1}^d \mathcal{H}_i$, where each $\mathcal{H}_i \simeq \mathbb{C}$. On $\mathbb{B}(\mathbb{C}\Omega \oplus \mathcal{H}_i) = M_2(\mathbb{C})$, let $\phi_i$ be the vector state for $\Omega$, in other words the $(1,1)$ entry of the matrix. In this algebra, let $p_i$ be a projection with $\phi[p_i] = \alpha_i$, and $p_i^\circ = p_i - \alpha_i$. Denote by $\mathcal{B}$ the subalgebra of $M_{d+1}(\mathbb{C})$ generated by $\{p_i^\circ : 1 \leq i \leq d\}$.

Note that $p_i^\circ = \beta_i(E_{1,i+1} + E_{i+1,1}) + \gamma_i E_{i+1,i+1}$, with $\beta_i \neq 0$. Then for $i \neq j$, $p_i^\circ p_j^\circ = \beta_i \beta_j E_{i+1,j+1}$. Therefore $E_{i+1,j+1} \in \mathcal{B}$ for $i \neq j$. Multiplying these, also $E_{i+1,i+1} \in \mathcal{B}$. Next, $p_i^\circ E_{i+1,i+1} = \beta_i E_{1,i+1} + \gamma_i E_{i+1,i+1}$, so also $E_{1,i+1} \in \mathcal{B}$, as is $E_{i+1,1}$. Multiplying these, also $E_{1,1} \in \mathcal{B}$. □

**Proposition 3.3.3.3.** *Let $\{\alpha_i : 1 \leq i \leq d\} \subset (0,1)$, and $\{p_i : 1 \leq i \leq d\}$ as in the preceding Corollary. In $(\Gamma_w(M_{d+1}(\mathbb{C}), \phi_{11}; t), \Psi)$, the von Neumann subalgebra generated by $X(p_i^\circ)$ is*

$$W^*(X(p_i^\circ) : 1 \leq i \leq d) \simeq \underset{1-\gamma}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma}{\mathbb{C}},$$

*where*

$$\gamma = \max\left\{1 - t\sum_{i=1}^d \frac{\alpha_i(1-\alpha_i)}{(1-2\alpha_i)^2}, 0\right\},$$

*and $x$ is chosen so that the free dimension is the sum of the free dimensions of $W^*(X(p_i^\circ))$.*

*Proof.* The argument is similar to Theorem 3.3.2.2.

Since we assume $\{p_i^\circ : 1 \leq i \leq d\}$ are Boolean independent, by Corollary 3.2.2.2, their corresponding $X(p_i^\circ)$ are freely independent with respect to $\Psi$, so

$$W^*(X(p_i^\circ)\,:\,1\le i\le d) = *_{i=1}^{d} W^*(X(p_i^\circ)).$$

By Theorem 3.3.1.1, we have,

$$W^*(X(p_1^\circ),\Psi) \simeq \underset{1-\gamma}{\mathcal{L}(\mathbb{F}_1)} \oplus \underset{\gamma_1}{\mathbb{C}}, \tag{3.7}$$

where $\gamma_1 = \max\left\{1 - t\frac{\alpha_1(1-\alpha_1)}{(1-2\alpha_1)^2}, 0\right\}$, which is nonzero if and only if $\frac{1}{\alpha_1(1-\alpha_1)} - 4 > t$. The left-hand side is positive if and only if $\alpha_1 \neq \frac{1}{2}$.

Applying Proposition 2.4 of [13], we have

$$W^*(X(p_1^\circ),\Psi) * W^*(X(p_2^\circ),\Psi) \simeq \underset{\gamma}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{1-\gamma}{\mathbb{C}},$$

where $\gamma_2 = \max\left\{1 - t\left(\frac{\alpha_1(1-\alpha_1)}{(1-2\alpha_1)^2} + \frac{\alpha_2(1-\alpha_2)}{(1-2\alpha_2)^2}\right), 0\right\}$.

Through repeated applications of Proposition 2.4 of [13], it is clear that

$$(\Gamma_a(\mathcal{B}_i,\phi;t),\Psi)_{i=1,\dots,d}^{*d} \simeq \underset{1-\gamma}{\mathcal{L}(\mathbb{F}_x)} \oplus \underset{\gamma}{\mathbb{C}},$$

where

$$\gamma = \max\left\{1 - t\sum_{i=1}^{d}\frac{\alpha_i(1-\alpha_i)}{(1-2\alpha_i)^2}, 0\right\}.$$

□

**Corollary 3.3.3.4.** *For $1\le i\le d$, let $p_i^\circ = \frac{1}{2}(E_{1,1+i} + E_{1+i,1})$. In $(\Gamma_w(M_{d+1}(\mathbb{C}),\phi_{11};t),\Phi)$, the von Neumann subalgebra generated by $X(p_i^\circ)$ is*

$$W^*(X(p_i^\circ) : 1\le i\le d) \simeq \begin{cases} \mathcal{L}(\mathbb{F}_d) & \text{if } t \ge \sqrt{d} \\ \mathcal{L}(\mathbb{F}_d)\oplus\mathbb{B}(\ell_2) & \text{otherwise.}\end{cases}$$

*Proof.* Rescaling a single $X(p^\circ, t)$ according to Corollary 3.3.1.2 gives a distribution with respect to $\Phi$ which matches that in Proposition 2.1 of [6] (with respect to what they call $\phi$), with $\theta$ taking the role played by $t$ in their densities. With respect to $\Psi$, we get the semi-circle distribution with radius $2\sqrt{\theta}$, which matches the distribution of Ricard's $t$-gaussian with respect to what they call $\psi$. Since we are taking conditionally free products, the joint distribution of multiple copies of $X(p^\circ, t)$ under the c-free product state is determined entirely by the individual distributions. Hence, their joint distribution matches that of Ricard's $t$-gaussians under their conditionally free product state. By Theorem 1.2.4.1, this is sufficient to conclude that our c-free product von Neumann algebra and theirs are isomorphic. Finally, a simple calculation converts their case cutoffs from terms of $\theta$ to $t$. $\square$

### 3.3.4 $\mathcal{B} = L^\infty[0, 2\pi]$

For this case, we will need the following surely well-known technical lemma, although we do not have a reference for it.

**Lemma 3.3.4.1.** *Let $H$ be a Hilbert space, and $\mathcal{A}$ a subalgebra of $\mathbb{B}(H)$, such that a vector $\Omega \in H$ is cyclic for $\mathcal{A}'$. Let $(x_n)_{n=1}^\infty \in \mathcal{A}$ be a sequence. If $x_n\Omega \to x\Omega$ for $x \in \mathcal{A}''$, then $x_n \to x$ in the WOT.*

*If the sequence is (norm) bounded, it converges $\sigma$-weakly.*

*Proof.* For $\xi \in \mathcal{H}$, take a bounded (by some $M > 0$) sequence $\{y_n\} \subset \mathcal{A}'$ such that $\|y_n\Omega - \xi\| \to 0$ as $n \to \infty$. By taking subsequences, we can assume that $\|y_n\Omega - \xi\| \leq \min\left\{\frac{1}{n\|x_n\|}, \frac{1}{n}\right\}$ Then

$$\begin{aligned}
|\langle x_n\xi - x\xi, \eta\rangle| &\leq |\langle x_n\xi - x_n y_n\Omega, \eta\rangle| + |\langle x_n y_n\Omega - x y_n\Omega, \eta\rangle| + |\langle x y_n\Omega - x\xi, \eta\rangle| \\
&\leq \|x_n\|\|y_n\Omega - \xi\|\|\eta\| + |\langle y_n(x_n\Omega - x\Omega), \eta\rangle| + \|x\|\|y_n\Omega - \xi\|\|\eta\| \\
&\leq \|x_n\|\|y_n\Omega - \xi\|\|\eta\| + M\|x_n\Omega - x\Omega\|\|\eta\| + \|x\|\|y_n\Omega - \xi\|\|\eta\|
\end{aligned}$$

all three of which converge to zero as $n \to \infty$.

The second claim follows from the fact that the weak and $\sigma$-weak topologies are equivalent on bounded subsets of $\mathcal{B}(\mathcal{H})$.

$\square$

The following identities all follow from the standard properties of an orthonormal basis.

**Lemma 3.3.4.2.** *Let $\mathcal{B}$ be commutative, and $\{e_k : k \in \mathbb{Z}\} \subset \mathcal{B}^\circ$ a family such that $\{e_k\Omega : k \in \mathbb{Z}_{\neq 0}\} \subset \mathcal{H}^\circ$ is an orthonormal basis and $e_k e_k^* = 1$. For example, we may take $\mathcal{B} = L^\infty[0,1)$, $\Omega = 1$, and $e_k(x) = e^{2\pi i k x}$ for $k \neq 0$. Then for any $f, g \in \mathcal{B}^\circ$, we have*

$$\frac{1}{N}\sum_{k=1}^{N} \langle e_k e_k^* \Omega, \Omega \rangle \, \Omega = \Omega, \tag{3.8}$$

$$\lim_{N\to\infty} \sum_{j,k=1}^{N} \langle e_k f\Omega, \Omega\rangle \langle \Omega, e_k g\Omega\rangle \langle \Omega, e_j f\Omega\rangle \langle e_j g\Omega, \Omega\rangle = |\langle f\Omega, g\Omega\rangle|^2, \tag{3.9}$$

$$\lim_{N\to\infty} \sum_{j,k=1}^{N} \langle e_k f\Omega, \Omega\rangle \langle \Omega, e_j f\Omega\rangle \langle e_j g\Omega, e_k g\Omega\rangle = \langle f f^*\Omega, g g^*\Omega\rangle, \tag{3.10}$$

$$\langle e_k f e_k^* \Omega, \Omega\rangle = 0, \tag{3.11}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j,k=1}^{N} \langle e_k f\Omega, e_j f\Omega\rangle \langle e_j g\Omega, e_k g\Omega\rangle = \langle f f^*\Omega, g g^*\Omega\rangle, \tag{3.12}$$

*and*

$$\lim_{N\to\infty} \sum_{j,k=1}^{N} \Big\langle \Big( e_k f e_k^*\Omega - \langle e_k^* f\Omega, \Omega\rangle e_k\Omega - \langle e_k f e_k^*\Omega, \Omega\rangle \Omega \Big) - f\Omega, \\ \Big( e_j f e_j^*\Omega - \big\langle e_j^* f\Omega, \Omega\big\rangle e_j\Omega - \big\langle e_j f e_j^*\Omega, \Omega\big\rangle \Omega \Big) - f\Omega \Big\rangle = \langle f\Omega, f\Omega\rangle. \tag{3.13}$$

Throughout this section, we will compute the limits of either

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k) X(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)$$

or

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k)X_r(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)$$

In either case, we have 9 possible, terms, involving pairs of $a^+, a^-, a^0$. The square of the sum of the terms of each type looks like

$$\frac{1}{N^2}\sum_{j,k=1}^{N} \left\langle a^\star(e_k)a^\dagger_\cdot(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega), a^\star(e_j)a^\dagger_\cdot(e_j^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)\right\rangle$$

and so has $N^2$ terms. For the 5 terms involving some $a^+$, we use orthogonality of $e_k$ to conclude that the sum in the norm has only $N$ terms (the only exceptions are the actions of $a^-a^+$ and $a^0a^+$ on $\Omega$). For the $a^-, a^-$ case we use the first identity in the lemma, for the $a^-, a^0$ case the second and the fourth, for the $a^0, a^0$ case the third and the fifth.

**Remark 3.3.4.3.** Note that the right operator is

$$\begin{aligned} X_r(f)(\xi_1 \otimes \ldots \otimes \xi_n) &= \xi_1 \otimes \ldots \otimes \xi_n \otimes (f\Omega - \phi[f]\Omega) \\ &\quad + \xi_1 \otimes \ldots \otimes \xi_{n-1} \otimes (\xi_n f - \langle \xi_n f, \Omega\rangle \Omega) + t\xi_1 \otimes \ldots \otimes \xi_{n-1} \langle \xi_n f, \Omega\rangle, \end{aligned}$$

$$X_r(f)(\xi_1) = \xi_1 \otimes (f\Omega - \phi[f]\Omega) + (\xi_1 f - \langle \xi_1 f, \Omega\rangle \Omega) + (1+t)\langle \xi_1 f, \Omega\rangle \Omega,$$

$$X_r(f)\Omega = f\Omega.$$

See the final property in Remark 3.1.0.4. Since $\mathcal{B}$ is commutative, $\phi$ is tracial.

**Theorem 3.3.4.4.** *Let $\mathcal{B} = L^\infty[0,1)$ and $t \geq 0$. Let $\phi$ be given by integration against a non-atomic probability measure $\mu$. Note that without loss of generality, we may take $\mu$ to be the Lebesgue measure. Then for $n \geq 2$ and $f_1, \ldots, f_n \in \mathcal{B}^\circ$, as $N \to \infty$, $\sigma$-weakly*

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k, t)W(f_1 \otimes \ldots \otimes f_n)X(e_k^*, t) \to 0,$$

*while*

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)X(f,t)X(e_k^*,t) \to X(f,t),$$

*and*

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)X(e_k^*,t) \to (1+t).$$

*Proof.* Clearly, each sequence $\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)W(f_1\otimes\ldots\otimes f_n)X(e_k^*,t)$ is bounded. Also, all the assumptions in Theorem 2.5.0.3 are satisfied, which implies in particular that the vacuum vector is cyclic for the commutant of $\Gamma_w(\mathcal{B},\phi;t)$. Therefore by Lemma 3.3.4.1, it suffices to show that

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)W(f_1\otimes\ldots\otimes f_n)X(e_k^*,t)\Omega \to 0,$$

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)X(f,t)X(e_k^*,t)\Omega \to f,$$

and

$$\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)X(e_k^*,t)\Omega \to (1+t)\Omega.$$

Note that

$$\begin{aligned}\sum_{k=1}^{N} X(e_k,t)W(f_1\otimes\ldots\otimes f_n)X(e_k^*,t)\Omega &= \sum_{k=1}^{N} X(e_k,t)W(f_1\otimes\ldots\otimes f_n)X_r(e_k^*,t)\Omega \\ &= \sum_{k=1}^{N} X(e_k,t)X_r(e_k^*,t)(f_1\otimes\ldots\otimes f_n).\end{aligned}$$

Decompose each $X(e_k,t) = e^+(e_k) + a^-(e_k) + a^0(e_k)$, and the same for $X_r(e_k)$. Then we have $9$ cases.

- $\frac{1}{N}\left\|\sum_{k=1}^{N} a^\star(e_k)a_r^+(e_k^*)(f_1\Omega\otimes\ldots\otimes f_n\Omega)\right\|_t^2 \to 0$, for $\star = +,-,0$ and $n \geq 1$, all follow from orthogonality between different $e_k$, since the expansion of the inner product will have coefficient $\frac{1}{N^2}$ but only $N$ uniformly bounded terms. The same argument gives the vanishing limit for $a^+(e_k)a_r^\star(e_k^*)$. For $n = 0$, the limits are also zero, with the single exception

$\frac{1}{N}\sum_{k=1}^{N} a^{-}(e_k)a_r^{+}(e_k^*)\Omega = (1+t)\Omega$ as seen in equation (3.8).

- $\frac{1}{N}\left\|\sum_{k=1}^{N} a^{-}(e_k)a_r^{-}(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)\right\|_t \to 0$ for $n \geq 2$ follows from equation (3.9), and for $n \leq 1$ by definition.

- $\frac{1}{N}\left\|\sum_{k=1}^{N} a^{-}(e_k)a_r^{0}(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)\right\|_t \to 0$ for $n \geq 2$ follows from equations (3.9) and (3.10). For $n = 1$ it follows from equation (3.11), and for $n = 0$ from the definition.

- $\frac{1}{N}\left\|\sum_{k=1}^{N} a^{0}(e_k)a_r^{-}(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)\right\|_t \to 0$ for $n \geq 2$ in the same way, for $n \leq 1$ by definition.

- $\frac{1}{N}\left\|\sum_{k=1}^{N} a^{0}(e_k)a_r^{0}(e_k^*)(f_1\Omega \otimes \ldots \otimes f_n\Omega)\right\|_t \to 0$ for $n \geq 2$ follows from equations (3.12) and (3.9). For $n = 1$, it follows from equation (3.13) that $\frac{1}{N}\sum_{k=1}^{N} a^{0}(e_k)a_r^{0}(e_k^*)f\Omega \to f\Omega$. For $n = 0$, the limit is zero by definition.

$\square$

The following corollary is a variation on Theorem 2.10 in [30].

**Corollary 3.3.4.5.** *Let $\mathcal{B} = L^\infty[0,1)$ and $t > 0$. The unique normal tracial state on the multinomial algebra $\Gamma_w(\mathcal{B}, \phi; t)$ is the vacuum state. Consequently this algebra is a type II$_1$-factor.*

*Proof.* Let $\tau$ be such a state. On the one hand, for $n \geq 1$, by the last part of Theorem 3.3.4.4,

$$\frac{1}{N}\tau\left[\sum_{k=1}^{N} X(e_k,t)W(f_1,\ldots,f_n)X(e_k^*,t)\right] = \tau\left[\left(\frac{1}{N}\sum_{k=1}^{N} X(e_k,t)X(e_k^*,t)\right)W(f_1,\ldots,f_n)\right]$$
$$\to (1+t)\tau\left[W(f_1,\ldots,f_n)\right].$$

On the other hand, by Theorem 3.3.4.4 again, for $n \geq 2$,

$$\frac{1}{N}\tau\left[\sum_{k=1}^{N} X(e_k,t)W(f_1,\ldots,f_n)X(e_k^*,t)\right] \to 0 = \langle W(f_1,\ldots,f_n)\Omega, \Omega\rangle.$$

Thus $\tau\left[W(f_1,\ldots,f_n)\right]=0$. Similarly, for $n=1$,

$$\frac{1}{N}\tau\left[\sum_{k=1}^{N}X(e_k,t)W(f_1)X(e_k^*,t)\right]\to\tau[W(f_1)].$$

For $t>0$ this implies that also $\tau[W(f_1)]=0=\langle W(f_1)\Omega,\Omega\rangle$. It follows that $\tau$ is the vacuum state. $\square$

Finally, we justify the choice of the coefficients in the annihilation operator in the main construction.

**Proposition 3.3.4.6.** *Define the operators $a^+$, $a^0$, $a^-$ as in the main construction, with the exception that $a^-(f)(g)=(1+s)a_\phi^-(f)(g)$. Suppose that $s\neq t$.*

*a. The vacuum state is not tracial.*

*b. Suppose in addition that $\mathcal{B}=L^\infty[0,1)$, with $\phi$ the usual integration. Then the $(s\neq t)$ version of $\Gamma_w(\mathcal{B},\phi;t)$ is the algebra of all bounded operators on the Fock space.*

*Proof.* (a) This follows from checking the conditions outlined in Theorem 2.5.0.3.

(b) The proof follows the general idea of [5]. Let $e_k=e^{2\pi ik\theta}$ as before. We next show that $\frac{1}{N}\sum_{k=1}^{N}X(e_k,t)X(e_k^*,t)$ converges to $S:=(1+t)I+(s-t)P_\Omega$ in SOT. The computation is similar to the proof of Theorem 3.3.4.4 using Lemma 3.3.4.2: $\frac{1}{N}\sum_{k=1}^{N}a^-(e_k)a^+(e_k^*)\to tI+(1+s-t)P_\Omega$, $\frac{1}{N}\sum_{k=1}^{N}a^0(e_k)a^0(e_k^*)\to I-P_\Omega$, and the rest of the limits are zero. It follows that $S$ is in $\Gamma_w(\mathcal{B},\phi;t)$. Next, $S^2-(1+t)S=(1+s)(s-t)P_\Omega$, which shows $P_\Omega\in\Gamma_w(\mathcal{B},\phi;t)$ since $s\neq t$. It is also easy to check that $\Omega$ is cyclic for $\Gamma_w(\mathcal{B},\phi;t)$. So by Proposition 2.4 of [5], the commutant of $\Gamma_w(\mathcal{B},\phi;t)$ is trivial. In other words, $\Gamma_w(\mathcal{B},\phi;t)$ is the algebra of all bounded operators on the Fock space. $\square$

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# APPENDIX A

# AN ALTERNATIVE CONSTRUCTION

In our initial studies of the three motivating constructions, we were faced with the decision of which construction to pursue: a Fock space over an operator algebra $\mathcal{B}$, or one over a Hilbert space $\mathcal{H}$. We chose the former, which we already discussed extensively in this work. Here, I will showcase the analogous results for the latter.

**Construction A.0.0.1.** Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space, and $\mathcal{H}$ its complexification, on which (to be consistent with the rest of the paper) we will denote the conjugation by $*$. Let $C : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ be a bounded linear map such that $C(f^* \otimes g^*) = C(f \otimes g)^*$, $I \otimes C$ and $C \otimes I$ commute, and $C + I \otimes I$ is positive. Let $\Lambda : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ be a bounded linear map such that

$$\langle g, \Lambda(b \otimes f)\rangle = \langle \Lambda(b^* \otimes g), f\rangle$$

and

$$C(\Lambda \otimes I) = (\Lambda \otimes I)(I \otimes C).$$

On the algebraic Fock space

$$\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n},$$

define the operator $K$ by

$$K_n = ((C + I^{\otimes 2}) \otimes I^{\otimes(n-2)})(I \otimes (C + I^{\otimes 2}) \otimes I^{\otimes(n-3)} \ldots (I^{\otimes(n-2)} \otimes (C + I^{\otimes 2}))$$

and $K = I \oplus I \oplus \bigoplus_{n=2}^{\infty} K_n$. Define the inner product on the Fock space by the linear extension of

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_k\rangle_C = \delta_{n=k} \langle f_1 \otimes \ldots \otimes f_n, K(g_1 \otimes \ldots \otimes g_k)\rangle ,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathcal{H}^{\otimes k}$. Denote the completion by $\mathcal{F}_C(\mathcal{H})$. Next, we define densely defined operators indexed by elements of $\mathcal{H}$ by the linear extension of

$$\begin{aligned}
\ell^*(f)(f_1\otimes\ldots\otimes f_n) &= \langle f_1, f^*\rangle\, f_2\otimes\ldots\otimes f_n,\\
a^+(f)(f_1\otimes\ldots\otimes f_n) &= f\otimes f_1\otimes\ldots\otimes f_n,\\
a^-(f)(f_1\otimes\ldots\otimes f_n) &= \ell^*(f)[(C+I)(f_1\otimes f_2)\otimes\ldots\otimes f_n]\\
&= \ell^*(f)[C(f_1\otimes f_2)\otimes\ldots\otimes f_n] + \langle f, f_1\rangle\, f_2\otimes\ldots\otimes f_n],\\
a^-(f)(f_1) &= \ell^*(f)(f_1) = \langle f_1, f^*\rangle\,\Omega,\\
a^0(f)(f_1\otimes\ldots\otimes f_n) &= \Lambda(f\otimes f_1)\otimes f_2\otimes\ldots\otimes f_n,\\
a^-(f)(\Omega) &= a^0(f)(\Omega) = 0.
\end{aligned}$$

We also define the vacuum state $\langle A\Omega,\Omega\rangle$ and

$$X(f) = a^+(f) + a^-(f) + a^0(f).$$

**Remark A.0.0.2.** Many properties of this construction parallel those of the body of the paper, and so are provided without proof.

**Lemma A.0.0.3.** *If $C + I^{\otimes 2}$ is injective, then the Fock space inner product in Construction A.0.0.1 is non-degenerate. In particular, this is the case if $\|C\| < 1$, or if $C + tI \geq 0$ for some $t < 1$.*

**Proposition A.0.0.4.** *For any $\vec{\xi}, \vec{\eta}$,*

$$\left\langle a^+(f)\vec{\xi},\vec{\eta}\right\rangle_C = \left\langle \vec{\xi}, a^-(f^*)\vec{\eta}\right\rangle_C \quad \textit{and} \quad \left\langle a^0(f)\vec{\xi},\vec{\eta}\right\rangle_C = \left\langle \vec{\xi}, a^0(f^*)\vec{\eta}\right\rangle_C.$$

The operators are bounded on $\mathcal{F}_C(\mathcal{H})$ under the default assumptions that $C : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ and $\Lambda : \mathcal{H}^{\otimes 2} \to \mathcal{H}$ are both bounded, and the inner product is non-degenerate if $C + I \otimes I$ is invertible.

**Remark A.0.0.5.** Example 2.1.1.1 fits in this setting with $C$ being the multiplication operator by $1 + w$. However, Example 2.1.3.1 does not, nor Example 2.1.4.1 except in the case where $\psi$ is a multiple of $\phi$. On the other hand, the examples below do not fit in the setting of Construction 2.1.0.1.

**Example A.0.0.6.** If $\{e_i : i \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}_{\mathbb{R}}$, we may define $C$ by $C(e_i \otimes e_j) = C_{ij} e_i \otimes e_j$. In this case it is also natural to define $\Lambda(e_i \otimes e_j) = \sum_k B_{ij}^k e_k$. This is an extension of the setting of [1] different from Example 2.1.2.1.

**Example A.0.0.7.** Let $\mathcal{H} = L^2([0,1], dx)$, and $\{e_i\}$ the Fourier basis. Then $C$ from the preceding example is a convolution operator on $L^2([0,1]^2, dx \otimes dx)$. That is,

$$C(f)(x,y) = \int f(u-x, v-y)\, d\nu(u,v).$$

for $\nu$ a positive measure on $[0,1]^2$. In this case, it is also natural to define $T : L^2([0,1]^2, dx \otimes dx) \to L^2([0,1], dx)$ by

$$T(f)(x) = \int f(u,v)\, d\rho_x(-u,-v).$$

for $\{\rho_x : x \in [0,1]\}$ a family of complex measures on $[0,1]^2$.

We next establish analogues of the results of Sections 2.2, 2.3, and 2.4.

**Definition A.0.0.8.** Let $W_0(f) = 1$, $W(f) = X(f)$,

$$W(f_1, f_2) = X(f_1)W(f_2) - W(\Lambda(f_1 \otimes f_2)) - \langle f_1, f_2 \rangle,$$

and for $n \geq 3$,

$$\begin{aligned}
&W(f_1, ..., f_n) \\
&= X(f_1)W(f_2, ..., f_n) - W(\Lambda(f_1 \otimes f_2), f_3, ..., f_n) - W(\ell^*(f_1)(C+I)(f_2 \otimes f_3), f_4, ..., f_n).
\end{aligned}$$

Denote $W(f) = 1 + \sum_{n=1}^{\infty} W_n(f)$.

Before stating convergence conditions, we have the following norm estimates, which easily follow from Lemma 2.4.0.1:

**Lemma A.0.0.9.** *For any $f \in \mathcal{H}$,*

$$\begin{aligned} \left\|a^+(f)\right\|_C &= \left\|a^-(f)\right\|_C \leq \sqrt{\|C+I\|}\,\|f\|\,, \text{ and} \\ \left\|a^0(f)\right\|_C &\leq \|\Lambda\|\;\|f\|\,. \end{aligned}$$

The proofs of the next two propositions follow in the same manner as those of Propositions 2.4.0.4 and 2.4.0.7, respectively.

**Proposition A.0.0.10.** *Let $f_1, \ldots, f_n \in \mathcal{H}$. Then there exist a universal constant $\alpha$ and a constant $K > 0$, dependent only on $\|C+I\|$ and $\|\Lambda\|$, such that for all $n$,*

$$\|W(f_1,\ldots,f_n)\| \leq \alpha^{n-1}\left(\sqrt{\|C+I\|}+\|\Lambda\|\right)^{n-1} K\,\|f_1\|\ldots\|f_n\|\,.$$

**Proposition A.0.0.11.** *Let $L = \max\{\sqrt{\|C+I\|}, \|\Lambda\|\}$. If $\|f\| \leq \frac{1}{4L}$, then the generating function $R'(f)$ converges, and thus the cumulant generating function does as well.*

**Proposition A.0.0.12.** *For each $n$-tuple of elements of $\mathcal{H}$, $R'[f_1,\ldots,f_n]$ is an operator on $\mathcal{H}$ determined by the relation*

$$R[X(f_0),\ldots,X(f_{n+1})] = \langle R'[f_1,\ldots,f_n]f_{n+1}, f_0^*\rangle\,.$$

*This operator is bounded via Lemma A.0.0.9 and the Riesz Representation Theorem.*

*The operators $R'_n[f] := R'[\underbrace{f,\ldots,f}_{n \text{ times}}]$ satisfy the recursion*

$$R'_n[f](g) = \sum_{i=0}^{n-2} R'_i[f]\ell^*(f)C\Big(R'_{n-i-2}[f]f\otimes g\Big) + R'_{n-1}[f]\Lambda(f\otimes g). \tag{A.1}$$

*Their generating function $R'(f) := \sum_{n=0}^{\infty} R'_n[f]$ (where $R'_0[f] = R'[\emptyset] = 1$) satisfies the equation*

$$R'(f)(g) = 1 + R'(f)\ell^*(f)C\big(R'(f)(f)\otimes g\big) + R'(f)\Lambda(f\otimes g). \tag{A.2}$$

Similarly to Theorem 2.5.0.3, one can give conditions under which the vacuum state is tracial on $\Gamma_{C,\Lambda}(\mathcal{H}) = W^*(X(f) : f \in \mathcal{H})$.

If $C = 0$, the setting of this section is precisely the same as in Theorem 2.5.1.2, so that theorem holds verbatim. That is, under the assumption that the vacuum state is tracial, one can define a product on $\mathcal{H}$ turning $\Gamma_{0,\Lambda}(\mathcal{H}) = W^*(X(f) : f \in \mathcal{H})$ into a free compound Poisson algebra.